CS_275 Automata and Formal Language Theory

Course Notes
Part III: Limits of Computation
Chapter III.1: Introduction

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Computability Theory

- A **computable function** is a function

$$f : A \to B$$

such that there is a *mechanical procedure* for computing for every $a \in A$ the result $f(a) \in B$.
- In this part (III) we study the **limits of computation**.
- The area studying computable functions and their limits is called **computability theory**.

Examples

- Define exp : $\mathbb{N} \to \mathbb{N}$, $\text{exp}(n) := 2^n$,

  where $\mathbb{N} = \{0, 1, 2, \ldots\}$.
  exp is *computable*.
  However, can we really compute
  exp(10000000000000000000000000000000000000000000)?
- Let String be the set of strings of ASCII symbols.

  Define a function check : String $\to \{\text{true}, \text{false}\}$ by

$$\text{check}(p) := \begin{cases} \text{true} & \text{if } p \text{ is a syntactically correct Java program,} \\ \text{false} & \text{otherwise.} \end{cases}$$

Is check computable or not?

Examples (Cont.)

- Define a function terminate : String $\to \{\text{true}, \text{false}\}$,

$$\text{terminate}(p) := \begin{cases} \text{true} & \text{if } p \text{ is a syntactically correct Java program with no input and outputs, which terminates;} \\ \text{false} & \text{otherwise.} \end{cases}$$

Is terminate *computable?*

Answer

- _____________________________
  (To be filled in during the lecture)

Problems in Computability

In order to understand and answer the questions we have to

- Give a precise definition of what *computable* means.
  - That will be a mathematical definition.
  - Such a notion is particularly important for showing that certain functions are *non-computable*.
- Then provide evidence that the definition of "*computable*" is the correct one.
  - That will be a philosophical argument.
- Develop methods for proving that certain functions are *computable or non-computable*.

Three Areas

Three areas are involved in computability theory.

- Mathematics.
  - Precise definition of computability.
  - Analysis of the concept.
- Philosophy.
  - Validation that notions found are the correct ones.
- Computer science.
  - Study of relationship between these concepts and computing in the real world.

Questions Related to The Above

- Given a function $f : A \to B$, which can be computed, can it be done *effectively*?
  (Complexity theory.)
- Can the task of deciding a given problem $P1$ be reduced to deciding another problem $P2$?
  (Reducibility theory).

More Advanced Questions

The following is beyond the scope of this module.

- Can the notion of *computability* be extended to computations on *infinite objects*
  (e.g. streams of data, real numbers,higher type operations)?
  (Higher and abstract computability theory).
- What is the relationship between *computing* (producing actions, data etc.) and *proving*?
  (Research by U. Berger and M. Seisenberger).

Idealisation

In computability theory, one usually abstracts from limitations on

- time and
- space.

A problem will be computable, if it can be solved on an *idealised computer*, even if it the computation would take longer than the life time of the universe.

Remark on Variables

- In this lecture I will often use i, j, k, l, m, n for variables denoting natural numbers.
- I will often use p, q and some others for variables denoting programs.
- I will use z for integers.
- Other letters might be used as well for variables.
- These conventions are not treated very strictly.
  - Especially when running out of letters.

History of Computability Theory

Gottfried Wilhelm von Leibniz (1646 – 1716)

- Built a first *mechanical calculator.*
- Was thinking about a machine for manipulating symbols in order to determine truth values of mathematical statements.
- Noticed that this requires the definition of a precise *formal language.*

History of Computability Theory

David Hilbert(1862 – 1943)

- Poses 1900 in his famous list "Mathematical Problems" as 10th problem to decide *Diophantine equations.*

Decision Problem

Hilbert (1928)

- Poses the *Entscheidungsproblem* (German for decision problem).
- The *decision problem* is the question, whether we can decide whether a formula in predicate logic is provable or not.
  - *Predicate logic* is the standard formalisation of logic with connectives $\wedge, \vee, \rightarrow, \neg$ and quantifiers $\forall$, $\exists$.
  - Predicate logic is "*sound and complete*".
    - This means that a a formula is provable if and only if it is valid (in all models).

Decision Problem

  - So the decidability of predicate logic is the question whether we can decide whether a formula is valid (in all models) or not.
  - If predicate logic were decidable, provability in mathematics would become trivial.
- "**Entscheidungsproblem**" became one of the few German words which have entered the English language.

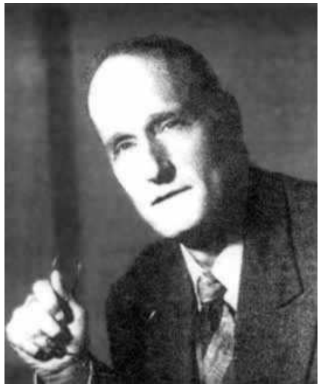
History of Computability Theory

- Gödel, Kleene, Post, Turing (1930s)
  Introduce different *models of computation* and prove that they all define the same class of computable functions.

History of Computability Theory

Kurt Gödel (1906 – 1978)
Introduced the (Herbrand-Gödel-) recursive functions
in his 1933 - 34 Princeton lectures.
Proved the famous incompleteness theorems now called first and second Gödel's Incompleteness Theorem

History of Computability Theory



Stephen Cole Kleene
(1909 – 1994)
Probably the most influential computability theoretician up to now.
Introduced the partial recursive functions.

History of Computability Theory

Emil Post
(1897 – 1954)
Introduced the Post problems.

History of Computability Theory

Alan Mathison Turing
(1912 – 1954)
Introduced the Turing machine.
Proved the undecidability
of the Turing-Halting problem.

Gödel's Incompleteness Theorem

- Gödel (1931) proves in his first incompleteness theorem:
  - Every reasonable primitive-recursive theory is incomplete, i.e. there is a formula s.t. neither the formula nor its negation is provable.
    - The theorem generalises to recursive i.e. computable theories as introduced later in this module.
    - For the moment it suffices to understand "recursive" informally as intuitively computable.
    - The notion "primitive-recursive" would be taught in an advanced module on computability theory.
      Primitive recursive functions form a restricted class of computable functions which contain all functions which are computable in a feasible way.

Gödel's Incompleteness Theorem

- Therefore no computable theory proves all true formulae.
- Therefore, it is undecidable whether a formula is true or not.
  - Otherwise, the theory consisting of all true formulae would be a complete computable theory.

Undecidability of the Decision Problem

- Church and Turing (1936) postulate that the models of computation established above define exactly the set of all computable functions (Church-Turing thesis, sometimes called Church's thesis).
- Both established independently undecidable problems and proved that the **decision problem** is **undecidable**, i.e. **unsolvable**.
  - Even for a **class of very simple formulae** we cannot decide the decision problem.

Undecidability of the Decision Problem

- Church shows the undecidability of equality in the λ-calculus.
- Turing shows the unsolvability (i.e. undecidability) of the **halting problem** just months after Church's result was obtained.
  - It is undecidable whether a Turing machine (and by the Church-Turing thesis equivalently any non-interactive computer program of a Turing-complete programming language) eventually stops.
  - That problem turns out to be the most important undecidable problem.

Undecidability and the Limit of Human Beings

- The undecidability of the Turing Halting Problem has similarities with limits in physics such as
  - Speed limited by speed of light.
  - Heisenberg's uncertainty principle.
- It limits what human beings can do
  - Even with the most advanced computers we cannot decide the Turing halting problem.

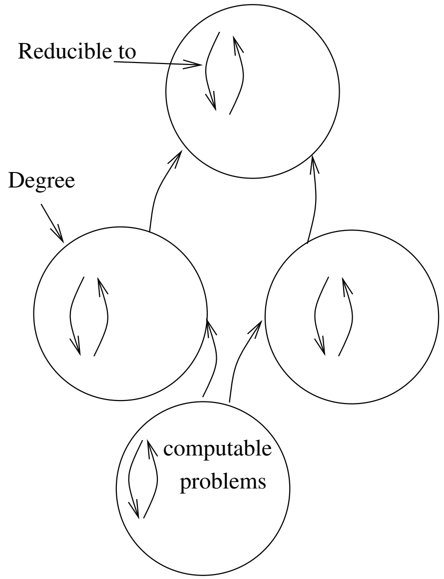
History of Computability Theory

Alonzo Church (1903 - 1995)

History of Computability Theory

- Post (1944) studies degrees of unsolvability. This is the birth of degree theory.
- In degree theory one divides problems into groups ("**degrees**") of problems, which are reducible to each other.
  - **Reducible** means essentially "relative computable".
- Degrees can be ordered by using reducibility as ordering.
- The question in degree theory is: what is the structure of degrees?

Degrees

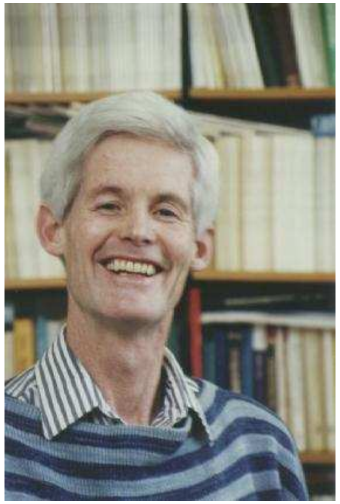


History of Computability Theory

Yuri Vladimirovich Matiyasevich ($*$ 1947)

- Solves 1970 Hilbert's 10th problem negatively: The solvability of Diophantine equations is undecidable.

History of Computability Theory



Stephen Cook(Toronto)

- Cook (1971) introduces the complexity classes P and NP and formulates the problem, whether $P \neq NP$.

Current State

- The problem $P \neq NP$ is still open. Complexity theory has become a big research area.
- Intensive study of computability on infinite objects (e.g. real numbers, higher type functionals) is carried out (e.g. U. Berger, J. Blanck and J. Tucker in Swansea).
- Computability on inductive and co-inductive data types is studied (e.g. A. Setzer, U. Berger in Swansea).
- Research on program synthesis from formal proofs (e.g. U. Berger and M. Seisenberger in Swansea).

Current State

- Concurrent and game-theoretic models of computation are developed (e.g. Prof. Moller in Swansea).
- Automata theory further developed.
- Alternative models of computation are studied (quantum computing, genetic algorithms).
- . . .

Name "Computability Theory"

- The original name was *recursion theory*, since the mathematical concept claimed to cover exactly the computable functions is called "recursive function".
- This name was changed to *computability theory* during the last 10 years.
- Many books still have the title "recursion theory".

Overview over Part III

Literature

- [Cu80] Cutland: *Computability*. Cambridge University Press, 1980.
  - Main text book.
- [Co90] Daniel E. Cohen: *Computability and Logic*. Ellis Horwood, 1990.
- [Su05] Thomas A. Sudkamp: *Languages and machines.* 3rd Edition, Addison-Wesley 2005.
- [BJB07] George S. Boolos, Richard C. Jeffrey, John Burgess: *Computability and logic.* 5th Ed. Cambridge Univ. Press, 2007
- [LP97] Lewis/Papadimitriou: *Elements of the Theory of Computation*. Prentice Hall, 2nd Edition, 2008.
- [Si05] Sipser: *Introduction to the Theory of Computation*. South-Western College Publishing. 3rd Edition, 2012.

Literature

- [Ma10] Martin: *Introduction to Languages and the Theory of Computation*. 4th Edition, McGraw Hill, 2010.
  - Criticised in Amazon Reviews. But several editions.
- [HMU07] John E. Hopcroft, R. Motwani and J. Ullman: *Introduction to Automata Theory, Languages, and Computation*. Addison Wesley, 3rd Ed, 2007.
  - Excellent book, mainly on automata theory context free grammars.
  - But covers Turing machines, decidability questions as well.

Literature

- [Ve06] Velleman: *How To Prove It*. Cambridge University Press, 2nd Edition, 2006.
  - Book on basic mathematics.
  - Useful if you need to fresh up your mathematical knowledge.
- [Gr99] Griffor (Ed.): *Handbook of Computability Theory*. North Holland, 1999.
  - Expensive. Postgraduate level.