



- ▶ -----  
(To be filled in during the lecture)

## Three Areas

Three areas are involved in computability theory.

- ▶ **Mathematics.**
  - ▶ Precise definition of computability.
  - ▶ Analysis of the concept.
- ▶ **Philosophy.**
  - ▶ Validation that notions found are the correct ones.
- ▶ **Computer science.**
  - ▶ Study of relationship between these concepts and computing in the real world.

## Problems in Computability

In order to understand and answer the questions we have to

- ▶ Give a precise definition of what *computable* means.
  - ▶ That will be a **mathematical definition**.
  - ▶ Such a notion is particularly important for showing that certain functions are *non-computable*.
- ▶ Then provide evidence that the definition of “*computable*” is the correct one.
  - ▶ That will be a **philosophical argument**.
- ▶ Develop methods for proving that certain functions are *computable or non-computable*.

## Questions Related to The Above

- ▶ Given a function  $f : A \rightarrow B$ , which can be computed, can it be done *effectively*?  
(**Complexity theory**.)
- ▶ Can the task of deciding a given problem  $P1$  be reduced to deciding another problem  $P2$ ?  
(**Reducibility theory**).

The following is beyond the scope of this module.

- ▶ Can the notion of *computability* be extended to computations on *infinite objects* (e.g. streams of data, real numbers, higher type operations)? (**Higher and abstract computability theory**).
- ▶ What is the relationship between *computing* (producing actions, data etc.) and *proving*? (Research by U. Berger and M. Seisenberger).

## Remark on Variables

- ▶ In the part on computability theory of this module I will often use *i, j, k, l, m, n* for variables denoting natural numbers.
- ▶ I will often use *p, q* and some others for variables denoting programs.
- ▶ I will use *z* for integers.
- ▶ Other letters might be used as well for variables.
- ▶ These conventions are not treated very strictly.
  - ▶ Especially when running out of letters.

## Idealisation

In computability theory, one usually abstracts from limitations on

- ▶ time and
- ▶ space.

A problem will be computable, if it can be solved on an *idealised computer*, even if the computation would take longer than the life time of the universe.

## History of Computability Theory



Gottfried Wilhelm  
von Leibniz (1646 – 1716)

- ▶ Built a first *mechanical calculator*.
- ▶ Was thinking about a machine for manipulating symbols in order to determine truth values of mathematical statements.
- ▶ Noticed that this requires the definition of a precise *formal language*.



### David Hilbert (1862 – 1943)

- ▶ Poses 1900 in his famous list “Mathematical Problems” as 10th problem to decide *Diophantine equations*.

- ▶ So the decidability of predicate logic is the question whether we can decide whether a formula is valid (in all models) or not.
- ▶ If predicate logic were decidable, provability in mathematics would become trivial.
- ▶ “**Entscheidungsproblem**” became one of the few German words which have entered the English language.

### Hilbert (1928)

- ▶ Poses the *Entscheidungsproblem* (German for decision problem).
- ▶ The *decision problem* is the question, whether we can decide whether a formula in predicate logic is provable or not.
  - ▶ *Predicate logic* is the standard formalisation of logic with connectives  $\wedge, \vee, \rightarrow, \neg$  and quantifiers  $\forall, \exists$ .
  - ▶ Predicate logic is “*sound and complete*”.
    - ▶ This means that a formula is provable if and only if it is valid (in all models).

- ▶ **Gödel, Kleene, Post, Turing (1930s)**  
Introduced different *models of computation* and proved that they all define the same class of computable functions.



**Kurt Gödel (1906 – 1978)**  
Introduced the (Herbrand-Gödel-) recursive functions in his 1933 - 34 Princeton lectures. Proved the famous incompleteness theorems now called first and second Gödel's Incompleteness Theorem



**Emil Post (1897 – 1954)**  
Introduced the Post problems.



**Stephen Cole Kleene (1909 – 1994)**  
Probably the most influential computability theoretician up to now. Introduced the partial recursive functions.



**Alan Mathison Turing (1912 – 1954)**  
Introduced the Turing machine. Proved the undecidability of the Turing-Halting problem.

## Gödel's Incompleteness Theorem

- ▶ **Gödel (1931)** proves in his first incompleteness theorem:
  - ▶ Every reasonable primitive-recursive theory is incomplete, i.e. there is a formula s.t. neither the formula nor its negation is provable.
    - ▶ The theorem generalises to recursive i.e. computable theories as introduced later in this module.
    - ▶ For the moment it suffices to understand “recursive” informally as intuitively computable.
    - ▶ The notion “primitive-recursive” would be taught in an advanced module on computability theory.  
Primitive recursive functions form a restricted class of computable functions which contain all functions which are computable in a feasible way.

## Undecidability of the Decision Problem

- ▶ **Church and Turing (1936)** postulate that the models of computation established above define exactly the set of all computable functions (Church-Turing thesis, sometimes called Church's thesis).
- ▶ Both established independently undecidable problems and proved that the **decision problem** is **undecidable**, i.e. **unsolvable**.
  - ▶ Even for a **class of very simple formulae** we cannot decide the decision problem.

## Gödel's Incompleteness Theorem

- ▶ Therefore no computable theory proves all true formulae.
- ▶ Therefore, it is undecidable whether a formula is true or not.
  - ▶ Otherwise, the theory consisting of all true formulae would be a complete computable theory.

## Undecidability of the Decision Problem

- ▶ Church shows the undecidability of equality in the  $\lambda$ -calculus.
- ▶ Turing shows the unsolvability (i.e. undecidability) of the **halting problem** just months after Church's result was obtained.
  - ▶ It is undecidable whether a Turing machine (and by the Church-Turing thesis equivalently any non-interactive computer program of a Turing-complete programming language) eventually stops.
  - ▶ That problem turns out to be the most important undecidable problem.

## Undecidability and the Limit of Human Beings

- ▶ The undecidability of the Turing Halting Problem has similarities with limits in physics such as
  - ▶ Speed limited by speed of light.
  - ▶ Heisenberg's uncertainty principle.
- ▶ It limits what human beings can do
  - ▶ Even with the most advanced computers we cannot decide the Turing halting problem.

## History of Computability Theory

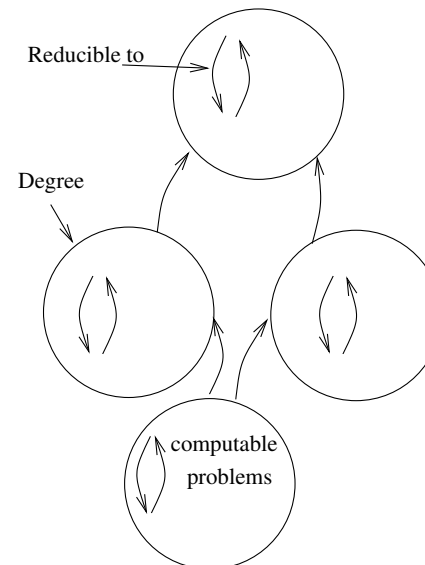
- ▶ **Post (1944)** studies degrees of unsolvability. This is the birth of degree theory.
- ▶ In degree theory one divides problems into groups ("degrees") of problems, which are reducible to each other.
  - ▶ Reducible means essentially "relative computable".
- ▶ Degrees can be ordered by using reducibility as ordering.
- ▶ The question in degree theory is: what is the structure of degrees?

## History of Computability Theory



Alonzo Church (1903 - 1995)

## Degrees



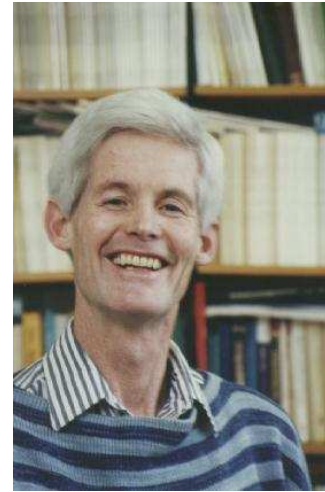


Yuri Vladimirovich  
Matiyasevich (\* 1947)

- ▶ Solves 1970 Hilbert's 10th problem negatively: The solvability of Diophantine equations is undecidable.

## Current State

- ▶ The problem  $P \neq NP$  is still open. Complexity theory has become a big research area.
- ▶ Intensive study of computability on infinite objects (e.g. real numbers, higher type functionals) is carried out (e.g. U. Berger, J. Blanck and J. Tucker in Swansea).
- ▶ Computability on inductive and co-inductive data types is studied (e.g. A. Setzer, U. Berger in Swansea).
- ▶ Research on program synthesis from formal proofs (e.g. U. Berger and M. Seisenberger in Swansea).



Stephen Cook(Toronto)

- ▶ Cook (1971) introduces the complexity classes  $P$  and  $NP$  and formulates the problem, whether  $P \neq NP$ .

## Current State

- ▶ Concurrent and game-theoretic models of computation are developed (e.g. Prof. Moller in Swansea).
- ▶ Automata theory further developed.
- ▶ Alternative models of computation are studied (quantum computing, genetic algorithms).
- ▶ ...



- ▶ The original name was *recursion theory*, since the mathematical concept claimed to cover exactly the computable functions is called “recursive function”.
- ▶ This name was changed to *computability theory* during the last 10 years.
- ▶ Many books still have the title “recursion theory”.

## Literature

- ▶ [Cu80] Cutland: *Computability*. Cambridge University Press, 1980.
  - ▶ Main text book.
- ▶ [Co90] Daniel E. Cohen: *Computability and Logic*. Ellis Horwood, 1990.
- ▶ [Su05] Thomas A. Sudkamp: *Languages and machines*. 3rd Edition, Addison-Wesley 2005.
- ▶ [BJB07] George S. Boolos, Richard C. Jeffrey, John Burgess: *Computability and logic*. 5th Ed. Cambridge Univ. Press, 2007
- ▶ [LP97] Lewis/Papadimitriou: *Elements of the Theory of Computation*. Prentice Hall, 2nd Edition, 2008.
- ▶ [Si05] Sipser: *Introduction to the Theory of Computation*. South-Western College Publishing. 3rd Edition, 2012.

1. [Introduction.](#)
2. [The Unlimited Register Machine \(URM\)](#)
3. [Turing machines and the undecidability of the halting problem.](#)

## Literature

- ▶ [Ma10] Martin: *Introduction to Languages and the Theory of Computation*. 4th Edition, McGraw Hill, 2010.
  - ▶ Criticised in Amazon Reviews. But several editions.
- ▶ [HMU07] John E. Hopcroft, R. Motwani and J. Ullman: *Introduction to Automata Theory, Languages, and Computation*. Addison Wesley, 3rd Ed, 2007.
  - ▶ Excellent book, mainly on automata theory context free grammars.
  - ▶ But covers Turing machines, decidability questions as well.

- ▶ [Ve06] Velleman: *How To Prove It*. Cambridge University Press, 2nd Edition, 2006.
  - ▶ Book on basic mathematics.
  - ▶ Useful if you need to fresh up your mathematical knowledge.
- ▶ [Gr99] Griffor (Ed.): *Handbook of Computability Theory*. North Holland, 1999.
  - ▶ Expensive. Postgraduate level.