CS_226 Computability Theory

Course Notes, Spring 2006

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Examples

- Define $exp : \mathbb{N} \to \mathbb{N}$, $\exp(n) := 2^n$,

  where $\mathbb{N} = \{0, 1, 2, \ldots\}$.
  exp is *computable*.
  However, can we really compute
  $\exp(100000000000000000000000000000000000000)$?
- Let $String$ be the set of strings of ASCII symbols.

  Define a function $check : \mathrm{String} \to \{0, 1\}$ by

$$\mathrm{check}(p) := \begin{cases} 1 & \text{if } p \text{ is a syntactically correct} \\ & \text{Java program,} \\ 0 & \text{otherwise.} \end{cases}$$

  Clearly, the function check is *computable*.

The Topic of Computability Theory

- A **computable function** is a function

$$f : A \to B$$

  such that there is a *mechanical procedure* for computing for every $a \in A$ the result $f(a) \in B$.
- **Computability theory** is the study of computable functions.
- In computablitity theory we explore the *limits* of the notion of computability.

Remark on Variables

- In this lecture I will often use i, j, k, l, m, n for variables denoting natural numbers.
- I will often use p, q and some others for variables denoting programs.
- I will use z for integers.
- Other letters might be used as well for variables.
- These conventions are not treated very strictly.
  - Especially when running out of letters.

Examples (Cont.)

- Define a function $terminate : \text{String} \to \{0, 1\}$,

$$\text{terminate}(p) := \begin{cases} 1 & \text{if } p \text{ is a syntactically correct} \\ & \text{Java program, which terminates,} \\ 0 & \text{otherwise.} \end{cases}$$

Is terminate *computable?*

Problems in Computability

In order to understand and answer the questions we have to

- Give a precise definition of what *computable* means.
  - That will be a **mathematical definition**.
  - Such a notion is particularly important for showing that certain functions are *non-computable*.
- Then provide evidence that the definition of "*computable*" is the correct one.
  - That will be a **philosophical argument**.
- Develop methods for proving that certain functions are *computable or non-computable*.

Examples (Cont.)

- Define a function $issortingfun : \text{String} \to \{0, 1\}$,

$$\text{issortingfun}(p) := \begin{cases} 1 & \text{if } p \text{ is a syntactically correct} \\ & \text{Java program, which has as input} \\ & \text{a list and returns a sorted list,} \\ 0 & \text{otherwise.} \end{cases}$$

Is issortingfun *computable?*

Three Areas

Three Areas are involved in computability theory.

- **Mathematics.**
  - Precise definition of computability.
  - Analysis of the concept.
- **Philosophy.**
  - Validation that notions found are the correct ones.
- **Computer science.**
  - Study of relationship between these concepts and computing in the real world.

Questions Related to The Above

- Given a function $f : A \to B$, which can be computed, can it be done *effectively*?
  (**Complexity theory**.)
- Can the task of deciding a given problem $P1$ be reduced to deciding another problem $P2$?
  (**Reducibility theory**).

Idealisation

In computability theory, one usually abstracts from limitations on

- time and
- space.

A problem will be computable, if it can be solved on an *idealised computer*, even if it the computation would take longer than the life time of the universe.

More Advanced Questions

The following is beyond the scope of this module.

- Can the notion of *computability* be extended to computations on *infinite objects*
  (e.g. streams of data, real numbers,higher type operations)?
  (**Higher and abstract computability theory**).
- What is the relationship between *computing* (producing actions, data etc.) and *proving*.

History of Computability Theory

Gottfried Wilhelm von Leibnitz (1646 – 1716)

- Built a first *mechanical calculator.*
- Was thinking about a machine for manipulating symbols in order to determine truth values of mathematical statements.
- Noticed that this requires the definition of a precise *formal language.*

History of Computability Theory

David Hilbert
(1862 – 1943)

- Poses 1900 in his famous list "Mathematical Problems" as 10th problem to decide *Diophantine equations.*

Entscheidungsproblem

- Hilbert refers to a theory developed by him for formalising mathematical proofs.
- Assumes that it is complete and sound, i.e. that it shows exactly all "true" formulae.
- Hilbert asks, whether there is an algorithm, which decides whether a mathematical formula is a consequence of his theory.
- Assuming his theory is complete and sound, such an algorithm would decide the *truth of all mathematical formulae.*
- The question, whether there is an algorithm for deciding the truth of mathematical formulae is later called the "*Entscheidungsproblem*".

History of Computability Theory

- **Hilbert (1928)**
  - Poses the "*Entscheidungsproblem*" (decision problem).

History of Computability Theory

- **Gödel, Kleene, Post, Turing (1930s)**
  Introduce different *models of computation* and prove that they all define the same class of computable functions.

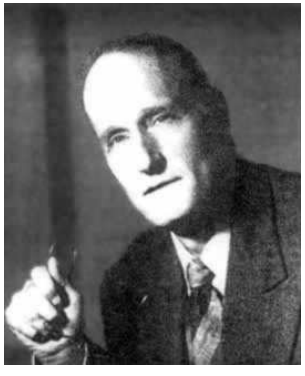
History of Computability Theory

Kurt Gödel (1906 – 1978)

History of Computability Theory

Emil Post
(1897 – 1954)
Introduced the Post problem

History of Computability Theory



Stephen Cole Kleene
(1909 – 1994)
Probably the most influential computability theoretist up to now. Introduced the partial recursive functions.

History of Computability Theory

Alan Mathison Turing
(1912 – 1954)
Introduced the Turing machine. Proved the undecidability of the Turing-Halting problem.

History of Computability Theory

- **Gödel (1931)** proves in his first incompleteness theorem:
  - Every reasonable recursive (i.e. computable) theory is incomplete, i.e. there is a formula s.t. neither the formula nor its negation is provable.
  - Therefore no such theory proves all true formulae.
- Recursive functions will be later shown to be the computable functions.
- Once this is established, it follows that the "Entscheidungsproblem" is unsolvable – an algorithm for deciding the truth of mathematical formulae would give rise to a complete and sound theory fulfilling Gödel's conditions.

History of Computability Theory

Alonzo Church (1903 - 1995)

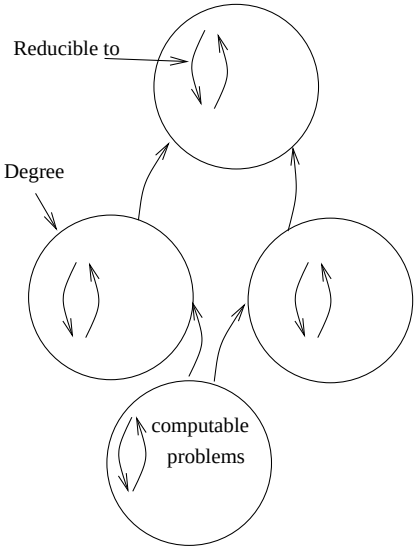
History of Computability Theory

- **Church, Turing (1936)** postulate that the models of computation established above define exactly the set of all computable functions (Church-Turing thesis).
- Both established undecidable problems and concluded that the **Entscheidungsproblem** is **unsolvable** even for a **class of very simple formulae**.
  - Church shows the undecidability of equality in the λ-calculus.
  - Turing shows the unsolvability of the halting problem.
    - It is undecidable whether a Turing machine (and by the Church-Turing thesis equivalently any non-interactive computer program) eventually stops.
    - That problem turns out to be the most important undecidable problem.

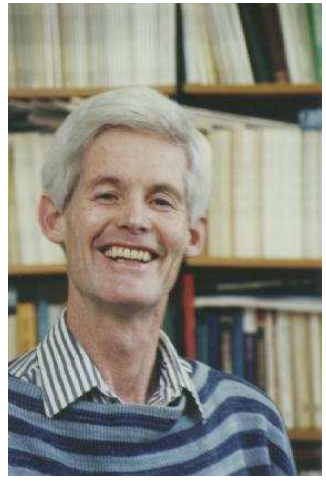
History of Computability Theory

- **Post (1944)** studies degrees of unsolvability. This is the birth of degree theory.
- In degree theory one devides problems into groups ("**degrees**") of problems, which are reducible to each other.
  - **Reducible** means essentially "relative computable".
- Degrees can be ordered by using reducibility as ordering.
- The question in degree theory is: what is the structure of degrees?

Degrees



History of Computability Theory



Stephen Cook(Toronto)

- **Cook (1971)** introduces the complexity classes **P** and **NP** and formulates the problem, whether **P** $\neq$ **NP**.

History of Computability Theory

Yuri Vladimirovich Matiyasevich ($*$ 1947)

- Solves 1970 Hilbert's 10th problem negatively: The solvability of Diophantine equations is undecidable.

Current State

- The problem **P** $\neq$ **NP** is still open. Complexity theory has become a big research area.
- Intensive study of computability on infinite objects (e.g. real numbers, higher type functionals) is carried out (e.g. U. Berger, Jens Blanck and J. Tucker in Swansea).
- Computability on inductive and co-inductive data types is studied.
- Research on program synthesis from formal proofs (e.g. U. Berger and M. Seisenberger in Swansea).

Current State

- Concurrent and game-theoretic models of computation are developed (e.g. Prof. Moller in Swansea).
- Automata theory further developed.
- Alternative models of computation are studied (quantum computing, genetic algorithms).
- . . .

Administrative Issues

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Name “Computability Theory”

- The original name was *recursion theory*, since the mathematical concept claimed to cover exactly the computable functions is called “recursive function”.
- This name was changed to *computability theory* during the last 10 years.
- Many books still have the title “recursion theory”.

Assessment:

- 80% Exam.
- 20% Coursework.

Course home page

- Located at
  `http://www.cs.swan.ac.uk/~csetzer/lectures/computability/05/index.html`
- There is an open version,
- and a password protected version.
- The password is _____.
- Errors in the notes will be corrected on the slides and noted on the list of errata.
- In order to reduce plagarism, coursework and solutions to coursework will **not** be made available in electronic form (e.g. on this web site).

Plan for this Module

8. Recursively enumerable predicates and the arithmetic hierarchy.

Plan for this Module

1. Introduction.
2. Encoding of data types into $\mathbb{N}$.
3. The Unlimited Register Machine (URM) and the halting problem.
4. Turing machines.
5. Algebraic view of computability.
6. Equivalence of models of computatation and the Church-Turing
7. The Recursion Theorem

Aims of this Module

- To become familiar with fundamental models of computation and the relationship between them.
- To develop an appreciation for the limits of computation and to learn techniques for recognising unsolvable or unfeasible computational problems.
- To understand the historic and philosophical background of computability theory.
- To be aware of the impact of the fundamental results of computability theory to areas of computer science such as software engineering and artificial intelligence.

Aims of this Module

- To understand the close connection between computability theory and logic.
- To be aware of recent concepts and advances in computability theory.
- To learn fundamental proving techniques like induction and diagonalisation.

Literature

- Martin: *Introduction to Languages and the Theory of Computation*. 3rd Edition, McGraw Hill, 2003.
  - Criticized in Amazon Reviews. But several editions.
- John E. Hopcroft, R. Motwani and J. Ullman: *Introduction to Automata Theory, Languages, and Computation*. Addison Wesley, 2nd Ed, 2001.
  - Excellent book, mainly on automata theory context free grammars.
  - But covers Turing machines, decidability questions as well.

Literature

- Cutland: *Computability*. Cambridge University Press, 1980.
  - Main text book.
- **Thomas A. Sudkamp:** *Languages and machines.* **Addison-Wesley 2006.**
- George S. Boolos, Richard C. Jeffrey, John Burgess: *Computability and logic.* 4th Ed. Cambridge Univ. Press, 2002
- Lewis/Papadimitriou: *Elements of the Theory of Computation*. Prentice Hall, 1981.
- Sipser: *Introduction to the Theory of Computation*. PWS Publishing, 1997.

Literature

- Velleman: *How To Prove It*. Cambridge University Press, 1994.
  - Book on basic mathematics.
  - Useful if you need to fresh up your mathematical knowledge.
- Griffor (Ed.): *Handbook of Computability Theory*. North Holland, 1999.
  - Expensive. Postgraduate level.