7. The Recursion Theorem

- Main result in this section:
  Kleene's Recursion Theorem.
  - Recursive functions are closed under a very general form of recursion.
- For proof we will use the **S-m-n-theorem**.
  - Used in many proofs in computability theory.

The S-m-n Theorem

$f : \mathbb{N}^{m+n} \xrightarrow{\sim} \mathbb{N}$ partial rec.
$g(\vec{x}) \simeq f(\vec{l}, \vec{x})$.

- So there exists a primitive recursive function S^m_n s.t.,
  - if $f = \{e\}^{m+n}$,
  - then $g = \{\mathsf{S}^m_n(e, \vec{l})\}^n$.

The S-m-n Theorem

- Assume $f : \mathbb{N}^{m+n} \xrightarrow{\sim} \mathbb{N}$ partial recursive.
- Fix the first m arguments (say $\vec{l} := l_0, \ldots, l_{m-1}$).
- Then we obtain a partial recursive function

$$g : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N}\ , \qquad g(\vec{x}) \simeq f(\vec{l}, \vec{x})\ .$$

- The S-m-n theorem expresses that we can compute a Kleene index of g
  - i.e. an e' s.t. $g = \{e'\}^n$

  from a Kleene index of f and $\vec{l}$ **primitive recursively**.

Theorem 7.1 (S-m-n Theorem)

- Assume $m, n \in \mathbb{N}$.
- There exists a primitive recursive function

$$\mathsf{S}^m_n : \mathbb{N}^{m+1} \to \mathbb{N}$$

s.t. for all $\vec{l} \in \mathbb{N}^m, \vec{x} \in \mathbb{N}^n$

$$\{\mathsf{S}^m_n(e, \vec{l})\}^n(\vec{x}) \simeq \{e\}^{m+n}(\vec{l}, \vec{x})\ .$$

Proof of S-m-n Theorem

- Let T be a TM encoded as e.
- A Turing machine T' corresponding to $\mathsf{S}^m_n(e,\vec{l})$ should be s.t.

$$\mathrm{T}'^{n}(\vec{x}) \simeq \mathrm{T}^{n+m}(\vec{l},\vec{x}) \enspace .$$

Proof of S-m-n Theorem

T is TM for e.
Want to define T' s.t. $\mathrm{T}'^{n}(\vec{x}) \simeq \mathrm{T}^{n+m}(\vec{l},\vec{x})$
T' can be defined as follows:

1. The initial configuration is:
   - $\vec{x}$ written on the tape,
   - head pointing to the left most bit:

$\cdots$	$\sqcup$	$\sqcup$	bin(x_0)	$\sqcup$	$\cdots$	$\sqcup$	bin(x_{n-1})	$\sqcup$	$\sqcup$	$\cdots$
			$\uparrow$							

Proof of S-m-n Theorem

T is TM for e.
Want to define T' s.t. $\mathrm{T}'^{n}(\vec{x}) \simeq \mathrm{T}^{n+m}(\vec{l},\vec{x})$
Initial configuration:

$\cdots$	$\sqcup$	$\sqcup$	bin(x_0)	$\sqcup$	$\cdots$	$\sqcup$	bin(x_{n-1})	$\sqcup$	$\sqcup$	$\cdots$
			$\uparrow$							

2. - T' writes first binary representation of $\vec{l} = l_0,\ldots,l_{n-1}$ in front of this.
   - terminates this step with the head pointing to the most significant bit of bin(l_0).

   So configuration after this step is:

bin(l_0)	$\sqcup$	$\cdots$	$\sqcup$	bin(l_{m-1})	$\sqcup$	bin(x_0)	$\sqcup$	$\cdots$	$\sqcup$	bir
$\uparrow$										

Proof of S-m-n Theorem

T is TM for e.
Want to define T' s.t. $\mathrm{T}'^{n}(\vec{x}) \simeq \mathrm{T}^{n+m}(\vec{l},\vec{x})$.
Configuration after first step:

bin(l_0)	$\sqcup$	$\cdots$	$\sqcup$	bin(l_{m-1})	$\sqcup$	bin(x_0)	$\sqcup$	$\cdots$	$\sqcup$	bir
$\uparrow$										

- Then T' runs T, starting in this configuration.
  It terminates, if T terminates.
  The result is

$$\simeq \mathrm{T}^{m+n}(\vec{l},\vec{x}) \enspace ,$$

  and we get therefore

$$\mathrm{T}'^{n}(\vec{x}) \simeq \mathrm{T}^{m+n}(\vec{l},\vec{x})$$

  as desired.

Proof of the S-m-n Theorem

T is TM for e.
T' is a TM s.t. ${\mathrm{T}'}^n(\vec{x}) \simeq \mathrm{T}^{n+m}(\vec{l}, \vec{x})$

- From a code for T one can now obtain a code for T' in a primitive recursive way.
- S^m_n is the corresponding function.
- The details will not be given in the lecture
  Jump over details

Proof of the S-m-n Theorem

- Assume, the terminating state of T'' has Gödel number (i.e. code) s, and that all other states have Gödel numbers $< s$.
- Then one appends to the instructions of T'' the instructions of T, but with the states shifted, so that the new initial state of T is the final state s of T'' (i.e. we add s to all the Gödel numbers of states occurring in T).
- This can be done as well primitive recursively.

Proof of the S-m-n Theorem

- A code for T' can be obtained from a code for T and from $\vec{l}$ as follows:
  - One takes a Turing machine T'', which writes the binary representations of

    $$\vec{l} = l_0, \ldots, l_{m-1}$$

    in front of its initial position (separated by a blank and with a blank at the end), and terminates at the left most bit.
  - It's a straightforward exercise to write a code for the instructions of such a Turing machine, depending on $\vec{l}$, and show that the function defining it is primitive recursive.

Proof of the S-m-n Theorem

- So a code for T'' can be defined primitive recursively depending on a code e for T and $\vec{l}$, and S^m_n is the primitive recursive function computing this. With this function it follows now that, if e is a code for a TM, then

  $$\{\mathsf{S}^m_n(e, \vec{l})\}^n(\vec{x}) \simeq \{e\}^{n+m}(\vec{l}, \vec{x}) \enspace .$$

  This equation holds, even if e is not a code for a TM: In this case $\{e\}^{m+n}$ interprets e as if it were the code for a valid TM T

Proof of the S-m-n Theorem

- (A code for such a valid TM is obtained by
  - deleting any instructions $\mathrm{encode}(q, a, q', a', D)$ in e s.t. there exists an instruction $\mathrm{encode}(q, a, q'', a'', D')$ occurring before it in the sequence e,
  - and by replacing all directions > 1 by $\ulcorner \mathsf{R} \urcorner = 1$.)

Notation

- We will in the following often omit the superscript n in $\{e\}^n(m_0, \ldots, m_{n-1})$.
  - I.e. we will write
$$\{e\}(m_0, \ldots, m_{n-1})$$
    instead of
$$\{e\}^n(m_0, \ldots, m_{n-1})$$
- Further $\{e\}$ not applied to arguments and without superscript means usually $\{e\}^1$.

Proof of the S-m-n Theorem

- $e' := \mathsf{S}^m_n(e, \vec{l})$ will have the same deficiencies as e, but when applying the Kleene-brackets, it will be interpreted as a TM T' obtained from e' in the same way as we obtained T from e, and therefore
$$\{e'\}^n(\vec{x}) \simeq \mathrm{T}'^n(\vec{x}) \simeq \mathrm{T}^{n+m}(\vec{l}, \vec{x}) \simeq \{e\}^{n+m}(\vec{l}, \vec{x}) \ .$$
  So we obtain the desired result in this case as well.

Notation

- We introduce as well some notations for dealing with $\vec{x}$, $\vec{y}$ etc.
  - $$\forall \vec{x} \in \mathbb{N}.\varphi(\vec{x})$$
    stands for
$$\forall x_1, \ldots, x_n \in \mathbb{N}.\varphi(x_1, \ldots, x_n)$$
    where the number of variables is implicit (and usually unimportant).
  - $$\exists \vec{x} \in \mathbb{N}.\varphi(\vec{x})$$
    is to be understood similarly.

Notation

- $$\{\vec{x} \in \mathbb{N}^n \mid \varphi(\vec{x})\}$$
  is to be understood as
  $$\{(x_1, \ldots, x_n) \in \mathbb{N}^n \mid \varphi(x_1, \ldots, x_n)\}$$
- $$\{(\vec{x}, y, z) \in \mathbb{N}^n \mid \varphi(\vec{x}, y, z)\}$$
  is to be understood as
  $$\{(x_1, \ldots, x_n, y, z) \in \mathbb{N}^n \mid \varphi(x_1, \ldots, x_n, y, z)\}$$
- Similar notations are to be understood analogously.

Examples

Kleene's Rec. Theorem: $\exists e.\forall \vec{x}.\{e\}^n(\vec{x}) \simeq f(e, \vec{x})$.

- There exists an e s.t.
  $$\{e\}(x) \simeq e + 1 \ .$$
  For showing this take in the Recursion Theorem $f(e, n) := e + 1$.
  Then
  $$\{e\}(x) \simeq f(e, x) \simeq e + 1 \ .$$

Kleene's Recursion Theorem

- Assume $f : \mathbb{N}^{n+1} \xrightarrow{\sim} \mathbb{N}$ partial recursive.
- Then there exists an $e \in \mathbb{N}$ s.t.
  $$\{e\}^n(\vec{x}) \simeq f(e, \vec{x}) \ .$$
  (Here $\vec{x} = x_0, \ldots, x_{n-1}$).

Examples

Kleene's Rec. Theorem: $\exists e.\forall \vec{x}.\{e\}^n(\vec{x}) \simeq f(e, \vec{x})$.

Remark:

- Such applications usually not very useful.
- Usually, when using the Rec. Theorem, one
  - doesn't use the index e directly,
  - but only the application of $\{e\}$ to arguments.

Examples

2. The function computing the **Fibonacci-numbers** fib is recursive.
   - (This is a weaker result than what we obtained above –
   - above we showed that it is even prim.-rec.)

Fibonacci Numbers

Show: Exists g rec.

s.t. $g(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ g(n \dot- 2) + g(n \dot- 1), & \text{otherwise.} \end{cases}$

Shown as follows: Define a recursive $f : \mathbb{N}^2 \to \mathbb{N}$ s.t.

$$f(e, n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \{e\}(n \dot- 2) + \{e\}(n \dot- 1), & \text{otherwise.} \end{cases}$$

Now let e be s.t.

$$\{e\}(n) \simeq f(e, n) \quad .$$

Then e fulfils the equations

$$\{e\}(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \{e\}(n \dot- 2) + \{e\}(n \dot- 1), & \text{otherwise.} \end{cases}$$

Fibonacci Numbers

Remember the defining equations for fib:

$$\begin{aligned} \mathsf{fib}(0) &= \mathsf{fib}(1) = 1 \quad , \\ \mathsf{fib}(n + 2) &= \mathsf{fib}(n) + \mathsf{fib}(n + 1) \quad . \end{aligned}$$

From these equations we obtain

$$\mathsf{fib}(n) = \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \mathsf{fib}(n \dot- 2) + \mathsf{fib}(n \dot- 1), & \text{otherwise.} \end{cases}$$

We show that there exists a recursive function $g : \mathbb{N} \to \mathbb{N}$, s.t.

$$g(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ g(n \dot- 2) + g(n \dot- 1), & \text{otherwise.} \end{cases}$$

Fibonacci Numbers

$$\{e\}(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ \{e\}(n \dot- 2) + \{e\}(n \dot- 1), & \text{otherwise.} \end{cases}$$

Let $g = \{e\}$.
Then we get

$$g(n) \simeq \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1, \\ g(n \dot- 2) + g(n \dot- 1), & \text{otherwise.} \end{cases}$$

These are the defining equations for fib.
One can show by induction on n that $g(n) = \mathsf{fib}(n)$ for all $n \in \mathbb{N}$.
Therefore fib is recursive.

General Applic. of Rec. Theorem

- Similarly, one can introduce arbitrary partial recursive functions g, where
  - $g(\vec{n})$ refers to arbitrary other values $g(\vec{m})$.

  This corresponds to the recursive definition of functions in programming.
  E.g. in Java one defines

```
public static int fib(int n){
  if (n == 0 || n == 1){
    return 1;}
  else{
    return fib(n-1) + fib(n-2);
  }
};
```


Example 3

As in general programming, recursively defined functions need not be total:

- There exists a partial recursive function $g : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ s.t.

$$g(x) \simeq g(x) + 1 \ .$$

- We get $g(x)\uparrow$.
- The definition of g corresponds to the following Java definition:

```
public static int g(int n){
  return g(n) + 1;
};
```

- When executing $g(x)$, Java loops.

Example 4

- There exists a partial recursive function $g : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ s.t.

$$g(x) \simeq g(x+1) + 1 \ .$$

  Note that that's a “black hole recursion”, which is not solvable by a total function.
- It is solved by $g(x)\uparrow$.
- Note that a recursion equation for a function f cannot always be solved by setting $f(x)\uparrow$.
  - E.g. the recursion equation for fib can't be solved by setting $\mathsf{fib}(n)\uparrow$.

Ackermann Function

- The Ackermann function is recursive:
  Remember the defining equations:

$$\begin{aligned} \mathsf{Ack}(0, y) &= y + 1 \ , \\ \mathsf{Ack}(x+1, 0) &= \mathsf{Ack}(x, 1) \ , \\ \mathsf{Ack}(x+1, y+1) &= \mathsf{Ack}(x, \mathsf{Ack}(x+1, y)) \ . \end{aligned}$$

- From this we obtain

$$\mathsf{Ack}(x, y) = \begin{cases} y + 1, & \text{if } x = 0, \\ \mathsf{Ack}(x \dot{-} 1, 1), & \text{if } x > 0 \text{ and } y = 0 \\ \mathsf{Ack}(x \dot{-} 1, \mathsf{Ack}(x, y \dot{-} 1)), & \text{otherwise.} \end{cases}$$

Ackermann Function

$$\mathsf{Ack}(x,y) = \begin{cases} y+1, & \text{if } x=0, \\ \mathsf{Ack}(x \dot{-} 1, 1), & \text{if } x>0 \text{ and } y=0, \\ \mathsf{Ack}(x \dot{-} 1, \mathsf{Ack}(x, y \dot{-} 1)), & \text{otherwise.} \end{cases}$$

- Define g partial recursive s.t.

$$g(x,y) \simeq \begin{cases} y+1, & \text{if } x=0, \\ g(x \dot{-} 1, 1), & \text{if } x>0 \wedge y=0, \\ g(x \dot{-} 1, g(x, y \dot{-} 1)), & \text{if } x>0 \wedge y>0. \end{cases}$$

- g fulfils the defining equations of Ack.
- Proof that $g(x,y) \simeq \mathsf{Ack}(x,y)$ follows by main induction on x, side-induction on y. The details will not be given in the lecture Jump over details.

Proof of Correctness of Ack

Show $g(x+1,y) = \mathsf{Ack}(x+1,y)$

- Base case $y=0$:

$$g(x+1,0) \simeq g(x,1) \overset{\text{Main-IH}}{=} \mathsf{Ack}(x,1) = \mathsf{Ack}(x+1,0) \ .$$

- Induction Step $y \to y+1$:

$$\begin{aligned} g(x+1,y+1) &\simeq g(x, g(x+1,y)) \\ &\overset{\text{Main-IH}}{\simeq} g(x, \mathsf{Ack}(x+1,y)) \\ &\overset{\text{Side-IH}}{\simeq} \mathsf{Ack}(x, \mathsf{Ack}(x+1,y)) \\ &= \mathsf{Ack}(x+1,y+1) \ . \end{aligned}$$

Proof of Correctness of Ack

- We show by induction on x that $g(x,y)$ is defined and equal to $\mathsf{Ack}(x,y)$ for all $x,y \in \mathbb{N}$:
  - Base case $x=0$.

$$g(0,y) = y+1 = \mathsf{Ack}(0,y) \ .$$

  - Induction Step $x \to x+1$. Assume

$$g(x,y) = \mathsf{Ack}(x,y) \ .$$

  We show

$$g(x+1,y) = \mathsf{Ack}(x+1,y)$$

  by side-induction on y:

Proof of the Rec. Theorem

Assume

$$f : \mathbb{N}^{n+1} \xrightarrow{\sim} \mathbb{N} \ .$$

We have to find an e s.t.

$$\forall \vec{x} \in \mathbb{N}. \{e\}^n(\vec{x}) \simeq f(e, \vec{x}) \ .$$

Idea:
Define $e = \mathsf{S}^1_n(e_1, e_2)$ for some (yet unknown) e_1, e_2.
We get:

$$\{e\}^n(\vec{x}) \simeq \{\mathsf{S}^1_n(e_1,e_2)\}^n(\vec{x}) \simeq \{e_1\}^{n+1}(e_2, \vec{x}) \ .$$

So we need to find e_1, e_2 s.t.

$$\forall \vec{x}. \underbrace{\{e_1\}^{n+1}(e_2,\vec{x})}_{\simeq \{e\}^n(\vec{x})} \simeq \underbrace{f(\mathsf{S}^1_n(e_1,e_2),\vec{x})}_{\simeq f(e,\vec{x})} \ .$$

Proof of Rec. Theorem

Need to solve:
$\forall \vec{x}.\{e_1\}^{n+1}(e_2, \vec{x}) \simeq f(\mathsf{S}^1_n(e_1, e_2), \vec{x})$.
Let e_1 s.t.

$$\{e_1\}^{n+1}(y, \vec{x}) \simeq f(\mathsf{S}^1_n(y, y), \vec{x}) \enspace .$$

So we need to solve

$$\underbrace{f(\mathsf{S}^1_n(e_1, e_2), \vec{x})}_{\simeq \{e_1\}^{n+1}(e_2, \vec{x})} \simeq f(\mathsf{S}^1_n(e_2, e_2), \vec{x}) \enspace .$$

Can be fulfilled by setting $e_2 := e_1$.
So, an index solving the problem is

$$e = \mathsf{S}^1_n(e_1, e_2) = \mathsf{S}^1_n(e_1, e_1) \enspace ,$$

and we are done.

Short Version of Proof

Let e_1 be s.t.

$$\{e_1\}^{n+1}(y, \vec{x}) \simeq f(\mathsf{S}^1_n(y, y), \vec{x}) \enspace .$$

Let $e := \mathsf{S}^1_n(e_1, e_1)$.
Then we have

$$\begin{array}{rcl}
\{e\}^n(\vec{x}) & \overset{e = \mathsf{S}^1_n(e_1, e_1)}{\simeq} & \{\mathsf{S}^1_n(e_1, e_1)\}^n(\vec{x}) \\
& \overset{\text{S-m-n theorem}}{\simeq} & \{e_1\}^{n+1}(e_1, \vec{x}) \\
& \overset{\text{Def of } e_1}{\simeq} & f(\mathsf{S}^1_n(e_1, e_1), \vec{x}) \\
& \overset{e = \mathsf{S}^1_n(e_1, e_1)}{\simeq} & f(e, \vec{x}) \enspace .
\end{array}$$