Sec. 8: Semi-Computable Predicates

We study $P \subseteq \mathbb{N}^n$, which are

- not decidable,
- but “half decidable”.

Official name is

- semi-decidable,
- or semi-computable.
- or recursively enumerable (r.e.).
Recursively enumerable stands for the definition based on the notion of partial recursive functions.

Semi-decidable or semi-computable stand for the definition based on an intuitive notion of “(partial) computable function”

Assuming the Church-Turing thesis, the two notions coincide.
Rec. Sets

Remember:

- A predicate $A$ is recursive, iff $\chi_A$ is recursive.
- So we have a “full” decision procedure:

\[
P(\vec{x}) \iff \chi_A(\vec{x}) = 1, \text{ i.e. answer yes },
\]
\[
\neg P(\vec{x}) \iff \chi_A(\vec{x}) = 0, \text{ i.e. answer no }.
\]
Semi-Decidable Sets

$P \subseteq \mathbb{N}^n$ will be semi-decidable, if there exists a partial recursive recursive function $f$ s.t.

$$P(\vec{x}) \iff f(\vec{x}) \downarrow.$$ 

- If $P(\vec{x})$ holds, we will eventually know it: the algorithm for computing $f$ will finally terminate, and then we know that $P(\vec{x})$ holds.
- If $P(\vec{x})$ doesn’t hold, then the algorithm computing $f$ will loop for ever, and we never get an answer.
Semi-Decidable Sets

So we have:

\[ P(\vec{x}) \iff f(\vec{x}) \downarrow \text{ i.e. answer yes ,} \]
\[ \neg P(\vec{x}) \iff f(\vec{x}) \uparrow \text{ i.e. no answer} \]

\[ \text{returned by } f . \]
Applications

One might think that semi-computable sets don’t occur in computing.

But they occur in many applications.

Examples are

- Checking whether a program terminates is semi-decidable.
- Checking whether a program in C++ is type correct is because of the template mechanism semi-decidable.
- In C++ compilers this problem is usually prevented by having a flag which limits the number of times templates are unfolded.
Applications

Examples (Cont.)

- Type checking in Agda (used in the module interactive theorem proving) is semi-decidable.
- Does in most applications not cause any problems.

Jump over next example
Applications

- Whether a statement is provable in many logical systems is semi-decidable.
- But even so this is semi-decidable, many search algorithms succeed in most practical cases.
- Often one can predict a certain time, after which normally the search algorithm should have returned an answer.
  - If the search algorithm hasn’t returned an answer after this time it is likely (but not guaranteed) that the statement is unprovable.
Def. 8.1 (Recursively Enumerable)

A predicate $A \subseteq \mathbb{N}^n$ is \textcolor{green}{recursively enumerable}, in short \textcolor{green}{r.e.}, if there exists a partial recursive function $f : \mathbb{N}^n \to \mathbb{N}$ s.t.

$$A = \text{dom}(f) .$$

Sometimes recursive predicates are as well called

- \textcolor{green}{semi-decidable} or
- \textcolor{green}{semi-computable} or
- \textcolor{green}{partially computable}.\"
Lemma 8.3

(a) Every recursive predicate is r.e.
(b) The halting problem, i.e.

\[ \text{Halt}^n(e, \vec{x}) : \iff \{e\}^n(\vec{x}) \downarrow , \]

is r.e., but not recursive.

The details given in the following and Theorem 8.4 will be omitted in this lecture Jump over details and Theorem 8.4.
Proof of Lemma 8.3

(a) Assume $A \subseteq \mathbb{N}^k$ is decidable.

Then

$$\mathbb{N}^k \setminus A$$

is recursive, therefore its characteristic function

$$\chi_{\mathbb{N}^k \setminus A}$$

is recursive as well.

Define

$$f : \mathbb{N}^k \xrightarrow{\sim} \mathbb{N},\ f(\vec{x}) :\xrightarrow{\sim} (\mu y.\chi_{\mathbb{N}^k \setminus A}(\vec{x}) \xrightarrow{\sim} 0) .$$

Note that $y$ doesn’t occur in the body of the $\mu$-expression.
Proof of Lemma 8.3

Then we have

If $A(\vec{x})$, then

$$\chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0,$$

so

$$f(\vec{x}) \simeq (\mu y. \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0) \simeq 0,$$

especially

$$f(\vec{x}) \downarrow.$$
Proof of Lemma 8.3

If \((\mathbb{N}^k \setminus A)(\bar{x})\), then

\[
\chi_{\mathbb{N}^k \setminus A}(\bar{x}) \simeq 1 ,
\]

so there exists no \(y\) s.t.

\[
\chi_{\mathbb{N}^k \setminus A}(\bar{x}) \simeq 0 .
\]

therefore

\[
f(\bar{x}) \simeq (\mu y. \chi_{\mathbb{N}^k \setminus A}(\bar{x}) \simeq 0) \simeq \bot ,
\]

especially

\[
f(\bar{x}) \uparrow .
\]
Proof of Lemma 8.3

So we get

\[ A(\bar{x}) \iff f(\bar{x}) \downarrow \iff \bar{x} \in \text{dom}(f), \]

\[ A = \text{dom}(f) \text{ is r.e.}. \]
Proof of Lemma 8.3

(b) We have

\[ \text{Halt}^n(e, \vec{x}) :\Leftrightarrow f_n(e, \vec{x}) \downarrow , \]

where \( f_n \) is partial recursive as in Sect. 5 s.t.

\[ \{e\}^n(\vec{x}) \simeq f_n(e, \vec{x}) . \]

So

\[ \text{Halt}^n = \text{dom}(f_n) \text{ is r.e.} . \]

We have seen above that \( \text{Halt}^n \) is non-computable, i.e. not recursive.

Jump over Theorem 8.4.
Theorem 8.4

There exist r.e. predicates

\[ W^n \subseteq \mathbb{N}^{n+1} \]

s.t., with

\[ W^n_e := \{ \vec{x} \in \mathbb{N}^n \mid W^n(e, \vec{x}) \} \]

we have the following:

- Each of the predicates \( W^n_e \subseteq \mathbb{N}^n \) is r.e.
- For each r.e. predicate \( P \subseteq \mathbb{N}^n \) there exists an \( e \in \mathbb{N} \) s.t. \( P = W^n_e \), i.e.

\[ \forall \vec{x} \in \mathbb{N}. P(\vec{x}) \iff W^n_e(\vec{x}) \]
Therefore, the r.e. sets $P \subseteq \mathbb{N}^n$ are exactly the sets $W_e^n$ for $e \in \mathbb{N}$. 
Remark on Theorem 8.4

- $W^n_e$ is therefore a **universal recursively enumerable sets**, which encodes all other recursively enumerable sets.

- The theorem means that we can assign to every recursively enumerable predicate $A$ a natural number, namely the $e$ s.t. $A = W^n_e$.
  - Each code denotes one predicate.
  - However, several numbers denote the same predicate:
    - there are $e, e'$ s.t. $e \neq e'$, but $W^n_e = W^n_{e'}$.
    - (Since there are $e \neq e'$ s.t. $\{e\}^n = \{e'\}^n$.

CS_226 Computability Theory, Lent Term 2008, Sect. 8
Proof Idea for Theorem 8.4

\[ W_e^n := \text{dom}(\{e\}^n) \].

If \( A \) is r.e., then \( A = \text{dom}(f) \) for some partial rec. \( f \).

Let \( f = \{e\}^n \).

Then \( A = W_e^n \).

The details given in the following will be omitted in the lecture. Jump over Details
Proof of Theorem 8.4

Let $f_n$ s.t.
\[
\forall e, \vec{n} \in \mathbb{N}. f_n(e, \vec{x}) \simeq \{e\}(\vec{x}) .
\]

Define
\[
W^n := \text{dom}(f_n).
\]

$W^n$ is r.e.

We have
\[
\vec{x} \in W^n_e \iff (e, \vec{x}) \in W^n \\
\iff f_n(e, \vec{x}) \downarrow \\
\iff \{e\}(\vec{x}) \downarrow \\
\iff \vec{x} \in \text{dom}(\{e\}^n).
\]
Proof of Theorem 8.4

Therefore

\[ W^n_e = \text{dom}(\{e\}^n) \]

\( W^n \) is r.e., since \( f_n \) is partial recursive.

Furthermore, we have for any set \( A \subseteq \mathbb{N}^n \)

\[ A \text{ is r.e.} \iff A = \text{dom}(f) \text{ for some partial recursive } f \]
\[ \iff A = \text{dom}(\{e\}^n) \text{ for some } e \in \mathbb{N} \]
\[ \iff A = W^n_e \text{ for some } e \in \mathbb{N}. \]

This shows the assertion.
Let $A \subseteq \mathbb{N}^n$. The following is equivalent:

(i) $A$ is r.e.

(ii) $A = \{\vec{x} | \exists y. R(\vec{x}, y)\}$

for some primitive recursive predicate $R$.

(iii) $A = \{\vec{x} | \exists y. R(\vec{x}, y)\}$

for some recursive predicate $R$.

(iv) $A = \{\vec{x} | \exists y. R(\vec{x}, y)\}$

for some recursively enumerable predicate $R$. 
Theorem 8.5

(i) $A$ is r.e.

(v) $A = \emptyset$ or

$$A = \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\}$$

for some primitive recursive functions

$$f_i : \mathbb{N} \to \mathbb{N}.$$

(vi) $A = \emptyset$ or

$$A = \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\}$$

for some recursive functions

$$f_i : \mathbb{N} \to \mathbb{N}.$$
Remark

We can summarise Theorem 8.5 as follows: There are 3 equivalent ways of defining that $A \subseteq \mathbb{N}^n$ is r.e.:

1. $A = \text{dom}(f)$ for some partial recursive $f$;
2. $A = \emptyset$ or $A$ is the image of primitive recursive/recursive functions $f_0, \ldots, f_{n-1}$;
3. $A = \{\bar{x} \mid \exists y. R(\bar{x}, y)\}$ for some primitive recursive/recursive/r.e. $R$. 
Remark, Case \( n = 1 \)

For \( A \subseteq \mathbb{N} \) the following is equivalent:

- \( A \) is r.e.
- \( A = \emptyset \) or \( A = \text{ran}(f) \) for some primitive recursive \( f : \mathbb{N} \rightarrow \mathbb{N} \).
- \( A = \emptyset \) or \( A = \text{ran}(f) \) for some recursive \( f : \mathbb{N} \rightarrow \mathbb{N} \).

Therefore \( A \subseteq \mathbb{N} \) is r.e., if

- \( A = \emptyset \)
- or there exists a (prim.-)rec. function \( f \), which enumerates all its elements.

This explains the name “recursively enumerable predicate”.

Skip Proof.
Proof Idea for Theorem 8.5:

(i) \rightarrow (ii):
Assume $A$ is r.e., $A = \text{dom}(f)$, for $f$ partial recursive.

\[ A(\vec{x}) \iff f(\vec{x}) \downarrow \]
\[ \iff \exists y. \text{the TM for computing } f(\vec{x}) \text{ terminates after } y \text{ steps} \]
\[ \iff \exists y. R(\vec{x}, y) \]
Proof Idea for Theorem 8.5:

(i) → (ii), Cont

where

\[ R(\vec{x}, y) \Leftrightarrow \text{the TM for comp. } f(\vec{x}) \text{ termin. after } y \text{ steps} . \]

\[ R \text{ is primitive recursive.} \]
Proof Ideas

(ii) $\rightarrow$ (v), special case $n = 1$:

Assume

$A = \{ x \in \mathbb{N} | \exists y. R(x, y) \}$ where $R$ is prim. rec.

$A \neq \emptyset$,

$y \in A$ fixed.

Define $f : \mathbb{N} \rightarrow \mathbb{N}$ recursive,

$$
f(x) = \begin{cases} 
\pi_0(x), & \text{if } R(\pi_0(x), \pi_1(x)), \\
y, & \text{otherwise.}
\end{cases}
$$

Then $A = \text{ran}(f)$. 
(v), (vi) → (i), special case \( n = 1 \):
Assume
\[
A = \text{ran}(f) ,
\]
where \( f \) is (prim.-)recursive.
Then
\[
A = \text{dom}(g) ,
\]
where
\[
g(x) \simeq (\mu y. f(y) = x) .
\]
g is partial recursive.

The full details will be omitted in the lecture.
Proof of Theorem 8.5

(i) → (ii):

(The actual predicate $R$ we will take will be slightly differently from that in the proof idea – it is technically easier to prove the theorem this way.)

If $A$ is r.e., then for some partial recursive function $f : \mathbb{N}^n \sim \mathbb{N}$ we have

$$A = \text{dom}(f) .$$

Let $f = \{e\}^n$.

By Kleene’s Normal Form Theorem there exist a primitive recursive function $U : \mathbb{N} \to \mathbb{N}$ and a primitive recursive predicate $T_n \subseteq \mathbb{N}^{n+1}$ s.t.

$$\{e\}^n(\vec{x}) \simeq U(\mu y. T_n(e, \vec{x}, y)) .$$
Proof of Theorem 8.5

(i) → (ii) (Cont.)

Therefore

\[ A(\vec{x}) \iff \vec{x} \in \text{dom}(f) \]
\[ \iff \vec{x} \in \text{dom}(\{e\}^n) \]
\[ \iff U(\mu y. T_n(e, \vec{x}, y)) \downarrow \]

U prim. rec., therefore total
\[ \iff \mu y. T_n(e, \vec{x}, y) \downarrow \]
\[ \iff \exists y. T_n(e, \vec{x}, y) \]
\[ \iff \exists y. R(\vec{x}, y) . \]

where

\[ R(\vec{x}, y) \iff T_n(e, \vec{x}, y) . \]
Proof of Theorem 8.5

(i) → (ii) (Cont.)

Now $R$ is primitive recursive, and

$$A = \{ \bar{x} \mid \exists y. R(\bar{x}, y) \}.$$
Proof of Theorem 8.5

(ii) $\rightarrow$ (iii): Trivial.

(iii) $\rightarrow$ (iv): By Lemma 8.3.
Proof of Theorem 8.5

(iv) $\rightarrow$ (ii):

Assume

$$A = \{\vec{x} \mid \exists y. R(\vec{x}, y)\} ,$$

where $R$ is r.e.

By "(i) $\rightarrow$ (ii)" there exists a primitive recursive predicate $S$ s.t.

$$R(\vec{x}, y) \Leftrightarrow \exists z. S(\vec{x}, y, z) .$$

Therefore

$$A = \{\vec{x} \mid \exists y. \exists z. S(\vec{x}, y, z)\}$$
$$= \{\vec{x} \mid \exists y. S(\vec{x}, \pi_0(y), \pi_1(y))\}$$
$$= \{\vec{x} \mid \exists y. R'(\vec{x}, y)\} ,$$
Proof of Theorem 8.5

((iv) $\rightarrow$ (ii), Cont.)

Here

$$R'((\vec{x}, y) :\iff S((\vec{x}, \pi_0(y), \pi_1(y)) \text{ is primitive recursive.}$$
Proof of Theorem 8.5

(ii) → (v):

Assume \( A \) is not empty and \( R \) is primitive recursive s.t.
\[
A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \}.
\]

Let \( \vec{z} = z_0, \ldots, z_{n-1} \) be some fixed elements s.t. \( A(\vec{z}) \) holds.

Define for \( i = 0, \ldots, n - 1 \)

\[
f_i(x) :=
\begin{cases}
\pi_i^{n+1}(x), & \text{if } R(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_n^{n+1}(x)), \\
z_i, & \text{otherwise}.
\end{cases}
\]

\( f_i \) are primitive recursive.
Proof of Theorem 8.5

(ii) → (v), Cont.

We show

\[ A = \{ (f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N} \} . \]
Proof of Theorem 8.5

((ii) → (v), Cont.)

“⊇”:
Assume \( x \in \mathbb{N} \), and show

\[
A(f_0(x), \ldots, f_{n-1}(x))
\]

If \( R(\pi_{0}^{n+1}(x), \pi_{1}^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_{n}^{n+1}(x)) \), then

\[
\exists z. R(\pi_{0}^{n+1}(x), \pi_{1}^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), z)
\]

therefore

\[
(\pi_{0}^{n+1}(x), \pi_{1}^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x)) \in A
\]

therefore

\[
A(f_0(x), \ldots, f_{n-1}(x))
\]
Proof of Theorem 8.5

((ii) $\rightarrow$ (v), Cont.)

("$\supseteq$", Cont.):

If $(\mathbb{N}^k \setminus R)(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_n^{n+1}(x))$, then

$$f_i(x) = z_i \ ,$$

therefore by $A(\vec{z})$

$$A(f_0(x), \ldots, f_{n-1}(x)) \ .$$

So in both cases we get that

$$A(f_0(x), \ldots, f_{n-1}(x)) \ ,$$

so

$$\{ (f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N} \} \subseteq A \ .$$
Proof of Theorem 8.5

((ii) $\rightarrow$ (v), Cont.)

“$\subseteq$”:

Assume $A(x_0, \ldots, x_{n-1})$,

and show

$$\exists z. (f_0(z) = x_0 \land \cdots \land f_{n-1}(z) = x_{n-1}) .$$

We have for some $y$

$$R(x_0, \ldots, x_{n-1}, y) .$$

Let

$$z = \pi^{n+1}(x_0, \ldots, x_{n-1}, y) .$$
Proof of Theorem 8.5

((ii) → (v), Cont.); (“⊆”, Cont)

Then we have

\[ x_i = \pi_{n+1}^i(z) \quad \text{and} \quad y = \pi_{n+1}^n(z) \],

therefore

\[ R(\pi_{n+1}^0(z), \pi_{n+1}^1(z), \ldots, \pi_{n+1}^{n-1}(z), \pi_{n+1}^n(z)) \],

therefore for \( i = 0, \ldots, n - 1 \)

\[ f_i(z) = \pi_{n+1}^i(z) = x_i \].
Proof of Theorem 8.5

\((\text{ii}) \rightarrow (\text{v}), \text{Cont.}); (\text{“}\subseteq\text{”, Cont})\)

therefore

\[
(x_0, \ldots, x_{n-1}) = (f_0(z), \ldots, f_{n-1}(z)) \\
\in \{ (f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N} \},
\]

and we have

\[
A \subseteq \{ (f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N} \}.
\]

Therefore we have shown

\[
A = \{ (f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N} \},
\]

and the assertion follows.
Proof of Theorem 8.5

(v) → (vi): Trivial.

(vi) → (i):
- If $A$ is empty, then $A$ is recursive, therefore r.e.
- Assume

$$A = \{ (f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N} \}.$$  

for some recursive functions $f_i$.
- Define

$$f : \mathbb{N}^n \sim \mathbb{N},$$

s.t.

$$f(x_0, \ldots, x_{n-1}) :\equiv \mu x. (f_0(x) \simeq x_0 \land \cdots \land f_{n-1}(x) \simeq x_{n-1}).$$
Proof of Theorem 8.5

((vi) $\rightarrow$ (i), Cont.)

$f$ can be written as

$$f(x_0, \ldots, x_{n-1}) :\simeq \mu x.\(((f_0(x) \div x_0) + (x_0 \div f_0(x))) + ((f_1(x) \div x_1) + (x_1 \div f_1(x))) + \cdots + ((f_{n-1}(x) \div x_{n-1}) + (x_{n-1} \div f_{n-1}(x))) \simeq 0\),$$

therefore $f$ is partial recursive.
Proof of Theorem 8.5

((vi) → (i), Cont.)

Furthermore, we have

\[ A(x_0, \ldots, x_{n-1}) \iff \exists x \in \mathbb{N}. x_0 = f_0(x) \land \cdots \land x_{n-1} = f_{n-1}(x) \]

\[ \iff f(x_0, \ldots, x_{n-1}) \downarrow , \]

therefore

\[ A = \text{dom}(f) \text{ is r.e.} \]
Theorem 8.6

\[ A \subseteq \mathbb{N}^k \text{ is recursive iff both } A \text{ and } \mathbb{N}^k \setminus A \text{ are r.e.} \]

Proof idea:

"⇒" is easy.

For "⇐": Assume

\[ A(\vec{x}) \iff \exists y. R(\vec{x}, y) \]
\[ (\mathbb{N}^k \setminus A)(\vec{x}) \iff \exists y. S(\vec{x}, y) \]

In order to decide \( A \), search simultaneously for a \( y \) s.t. \( R(\vec{x}, y) \) and for a \( y \) s.t. \( S(\vec{x}, y) \) holds.

If we find a \( y \) s.t. \( R(\vec{x}, y) \) holds, then \( A(\vec{x}) \) holds.

If we find a \( y \) s.t. \( S(\vec{x}, y) \) holds, then \( \neg A(\vec{x}) \) holds.

The details of the proof will be omitted in this lecture.
Proof of Theorem 8.6, “⇒”

If $A$ is recursive, then both $A$ and $\mathbb{N}^k \setminus A$ are recursive, therefore as well r.e.
Proof of Theorem 8.6, “⇐”

Assume $A$, $\mathbb{N}^k \setminus A$ are r.e.

Then there exist primitive recursive predicates $R$ and $S$ s.t.

$$A = \{ \vec{x} | \exists y. R(\vec{x}, y) \},$$
$$\mathbb{N}^k \setminus A = \{ \vec{x} | \exists y. S(\vec{x}, y) \}.$$
Proof of Theorem 8.6, “⇐”

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ \mathbb{N}^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} . \]

By

\[ A \cup (\mathbb{N}^k \setminus A) = \mathbb{N}^k , \]

it follows

\[ \forall \vec{x}. ((\exists y. R(\vec{x}, y)) \lor (\exists y. S(\vec{x}, y))) , \]

therefore as well

\[ \forall \vec{x}. \exists y. (R(\vec{x}, y) \lor S(\vec{x}, y)) . \quad (\ast) \]
Proof of Theorem 8.6, “⇐”

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ \mathbb{N}^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} , \]
\[ \forall \vec{x}. \exists y. (R(\vec{x}, y) \lor S(\vec{x}, y)) . \quad (*) \]

- Define

\[ h : \mathbb{N}^n \to \mathbb{N} , \quad h(\vec{x}) := \mu y. (R(\vec{x}, y) \lor S(\vec{x}, y)) . \]

- \( h \) is partial recursive.
- By (*) we have \( h \) is total, so \( h \) is recursive.
- We show

\[ A(\vec{x}) \iff R(\vec{x}, h(\vec{x})) . \]
Proof of Theorem 8.6, “⇐”

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} \quad \land \quad \mathbb{N}^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} \]

\[ h(\vec{x}) := \mu y. (R(\vec{x}, y) \lor S(\vec{x}, y)) \]

Show \( A(\vec{x}) \iff R(\vec{x}, h(\vec{x})) \).

- If \( A(\vec{x}) \) then
  \[ \exists y. R(\vec{x}, y) \]
  and
  \[ \vec{x} \notin (\mathbb{N}^k \setminus A) \]
  therefore
  \[ \neg \exists y. S(\vec{x}, y) \]
  Therefore we have for the \( y \) found by \( h(\vec{x}) \) that \( R(\vec{x}, y) \) holds, i.e.
  \[ R(\vec{x}, h(\vec{x})) \]
Proof of Theorem 8.6, “⇐”

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ \mathbb{N}^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} , \]
\[ h(\vec{x}) := \mu y. (R(\vec{x}, y) \lor S(\vec{x}, y)) , \]

Show \( A(\vec{x}) \Leftrightarrow R(\vec{x}, h(\vec{x})) \).

On the other hand, if \( R(\vec{x}, h(\vec{x})) \) holds then

\[ \exists y. R(\vec{x}, y) , \]

therefore

\[ A(\vec{x}) . \]

Therefore

\[ A = \{ \vec{x} \mid R(\vec{x}, h(\vec{x})) \} \text{ is recursive.} \]
Theorem 8.7

Let \( f : \mathbb{N}^n \sim \mathbb{N} \).
Then
\[
\text{\( f \) is partial recursive} \iff \text{\( G_f \) is r.e.} \ .
\]

Proof idea for “⇐”:
Assume \( R \) primitive recursive s.t.
\[
G_f(\vec{x}, y) \iff \exists z. R(\vec{x}, y, z) \ .
\]
In order to compute \( f(\vec{x}) \), search for a \( y \) s.t. \( R(\vec{x}, \pi_0(y), \pi_1(y)) \) holds.
\( f(\vec{x}) \) will be the first projection of this \( y \).
The details of the proof will be omitted in this lecture.
Jump over details
Proof of Theorem 8.7, “⇒”

Assume \( f \) is partial recursive.

Then \( f = \{e\}^n \) for some \( e \in \mathbb{N} \).

By Kleene’s Normal Form Theorem we have

\[
f(\vec{x}) \simeq U(\mu y. T_n(\vec{x}, y)) ,
\]

for some primitive recursive relation

\[
T_n \subseteq \mathbb{N}^{n+1}
\]

and some primitive recursive function

\[
U : \mathbb{N} \rightarrow \mathbb{N}.
\]
Proof of Theorem 8.7, “⇒”

\[ f(\vec{x}) \simeq U(\mu y. T_n(\vec{x}, y)) \cdot \]

Therefore

\[
\begin{align*}
(\vec{x}, y) \in G_f & \iff (f(\vec{x}) \simeq y) \\
& \iff \exists z. (T_n(\vec{x}, z) \land \\
& \quad (\forall z' < z. \neg T_n(\vec{x}, z')) \\
& \quad \land U(z) = y),
\end{align*}
\]

Therefore \( G_f \) is r.e.
Proof of Theorem 8.7, “⇐”

If $G_f$ is r.e., then there exists a primitive recursive predicate $R$ s.t.

$$ f(\vec{x}) \simeq y \iff (\vec{x}, y) \in G_f \iff \exists z. R(\vec{x}, y, z) . $$

Therefore for any $z$ s.t. $R(\vec{x}, \pi_0(z), \pi_1(z))$ holds we have that

$$ f(\vec{x}) \simeq \pi_0(z) . $$

Therefore

$$ f(\vec{x}) \simeq \pi_0(\mu u. R(\vec{x}, \pi_0(u), \pi_1(u))) , $$

$f$ is partial recursive.
Lemma 8.8

The recursively enumerable sets are closed under:

(a) **Union:**
If $A, B \subseteq \mathbb{N}^n$ are r.e., so is $A \cup B$.

(b) **Intersection:**
If $A, B \subseteq \mathbb{N}^n$ are r.e., so is $A \cap B$.

(c) **Substitution by recursive functions:**
If $A \subseteq \mathbb{N}^n$ is r.e., $f_i : \mathbb{N}^k \to \mathbb{N}$ are recursive for $i = 0, \ldots, n$, so is

$$C := \{ \bar{y} \in \mathbb{N}^k \mid A(f_0(\bar{y}), \ldots, f_{n-1}(\bar{y})) \}.$$
Lemma 8.8

(d) (Unbounded) existential quantification:
If \( D \subseteq \mathbb{N}^{n+1} \) is r.e., so is
\[
E := \{ \bar{x} \in \mathbb{N}^n \mid \exists y. D(\bar{x}, y) \}.
\]

(e) Bounded universal quantification:
If \( D \subseteq \mathbb{N}^{n+1} \) is r.e., so is
\[
F := \{ (\bar{x}, z) \in \mathbb{N}^{n+1} \mid \forall y < z. D(\bar{x}, z) \}.
\]

The details of the proof will be omitted in this lecture.
Jump over details
Proof of Lemma 8.8

Let $A, B \subseteq \mathbb{N}^n$ be r.e.

Then there exist primitive recursive relations $R, S$ s.t.

$$A = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. R(\vec{x}, y) \} ,$$

$$B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. S(\vec{x}, y) \} .$$
Proof of Lemma 8.8 (a), (b)

\[ A = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. R(\vec{x}, y) \} \, , \]
\[ B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. S(\vec{x}, y) \} \, . \]

One can easily see that

\[ A \cup B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. (R(\vec{x}, y) \lor S(\vec{x}, y)) \} \, , \]
\[ A \cap B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. (R(\vec{x}, \pi_0(y)) \land S(\vec{x}, \pi_1(y))) \} \, . \]

therefore \( A \cup B \) and \( A \cap B \) are r.e.
Proof of Lemma 8.8 (c)

\[ A = \{ \overrightarrow{x} \in \mathbb{N}^n \mid \exists y. R(\overrightarrow{x}, y) \} , \]
\[ B = \{ \overrightarrow{x} \in \mathbb{N}^n \mid \exists y. S(\overrightarrow{x}, y) \} . \]

- Assume \( A \subseteq \mathbb{N}^n \) is r.e., \( f_i : \mathbb{N}^k \rightarrow \mathbb{N} \) are recursive for \( i = 0, \ldots, n. \)

- Need to show that

\[ C := \{ (\overrightarrow{y} \in \mathbb{N}^k \mid A(f_0(\overrightarrow{y}), \ldots, f_{n-1}(\overrightarrow{y})) \} . \]

is r.e.

- Follows by

\[ C = \{ \overrightarrow{y} \mid A(f_0(\overrightarrow{y}), \ldots, f_{n-1}(\overrightarrow{y})) \} = \{ \overrightarrow{y} \mid \exists z. R(f_0(\overrightarrow{y}), \ldots, f_{n-1}(\overrightarrow{y}), z) \} \text{ is r.e.} \]
Proof of Lemma 8.8 (d), (e)

(d) follows from Theorem 8.5.

(e):

Assume \( T \) is a primitive recursive predicate s.t.

\[
D = \{(\vec{x}, y) \in \mathbb{N}^{n+1} | \exists z. T(\vec{x}, y, z)\}
\]

Then we get

\[
F = \{(\vec{x}, y) | \forall y' < y.D(\vec{x}, y')\} \\
= \{(\vec{x}, y) | \forall y' < y. \exists z. T(\vec{x}, y', z)\} \\
= \{(\vec{x}, y) | \exists z. \forall y' < y. T(\vec{x}, y', (z)_{y'})\}
\]

is r.e.,

where in the last line we used that

\[
\{(\vec{x}, z) | \forall y' < y. T(\vec{x}, y', (z)_{y'})\}
\]

is primitive recursive.
Lemma 8.9

The r.e. predicates are not closed under complement:
There exists an r.e. predicate \( A \subseteq \mathbb{N}^n \) s.t. \( \mathbb{N}^n \setminus A \) is not r.e.

Proof:

- \( \text{Halt}^n \) is r.e.
- \( \mathbb{N}^n \setminus \text{Halt}^n \) is not r.e.
  - Otherwise by Theorem 8.6 \( \text{Halt}^n \) would be recursive.
  - But by Lemma 8.3. (b) \( \text{Halt}^n \) is not recursive.