Sec. 8: Semi-Computable Predicates

- We study $P \subseteq \mathbb{N}^n$, which are
  - not decidable,
  - but “half decidable”.
- Official name is
  - semi-decidable,
  - or semi-computable.
  - or recursively enumerable (r.e.).

Rec. Sets

Remember:
- A predicate $A$ is recursive, iff $\chi_A$ is recursive.
- So we have a “full” decision procedure:
  $$P(\bar{x}) \Leftrightarrow \chi_A(\bar{x}) = 1, \text{ i.e. answer yes },$$
  $$\neg P(\bar{x}) \Leftrightarrow \chi_A(\bar{x}) = 0, \text{ i.e. answer no }.$$

Rec. enum. vs. semi-decidable

- **Recursively enumerable** stands for the definition based on the notion of partial recursive functions.
- **Semi-decidable** or **semi-computable** stand for the definition based on an intuitive notion of “(partial) computable function”
- Assuming the **Church-Turing thesis**, the two notions coincide.

Semi-Decidable Sets

$P \subseteq \mathbb{N}^n$ will be semi-decidable, if there exists a partial recursive recursive function $f$ s.t.

$$P(\bar{x}) \Leftrightarrow f(\bar{x}) \downarrow.$$

- If $P(\bar{x})$ holds, we will eventually know it: the algorithm for computing $f$ will finally terminate, and then we know that $P(\bar{x})$ holds.
- If $P(\bar{x})$ doesn’t hold, then the algorithm computing $f$ will loop for ever, and we never get an answer.
Semi-Decidable Sets

So we have:

\[ P(\vec{x}) \iff f(\vec{x}) \downarrow \text{ i.e. answer yes } , \]
\[ \neg P(\vec{x}) \iff f(\vec{x}) \uparrow \text{ i.e. no answer returned by } f . \]

Applications

- One might think that semi-computable sets don’t occur in computing.
- But they occur in many applications.
- **Examples** are
  - Checking whether a program terminates is semi-decidable.
  - Checking whether a program in C++ is type correct is because of the template mechanism semi-decidable.
  - In C++ compilers this problem is usually prevented by having a flag which limits the number of times templates are unfolded.

Applications (Cont.)

- Type checking in Agda (used in the module interactive theorem proving) is semi-decidable.
- Does in most applications not cause any problems. Jump over next example

Whether a statement is provable in many logical systems is semi-decidable.
- But even so this is semi-decidable, many search algorithm succeed in most practical cases.
- Often one can predict a certain time, after which normally the search algorithm should have returned an answer.
  - If the search algorithm hasn’t returned an answer after this time it is likely (but not guaranteed) that the statement is unprovable.
**Def. 8.1 (Recursively Enumerable)**

A predicate \( A \subseteq \mathbb{N}^n \) is **recursively enumerable**, in short **r.e.**, if there exists a partial recursive function \( f : \mathbb{N}^n \xrightarrow{\sim} \mathbb{N} \) s.t.

\[
A = \text{dom}(f).
\]

Sometimes recursive predicates are as well called
- **semi-decidable** or
- **semi-computable** or
- **partially computable**.

**Lemma 8.3**

(a) Every recursive predicate is r.e.

(b) The **halting problem**, i.e.

\[
\text{Halt}^n(e, \vec{x}) :\iff \{e\}^n(\vec{x}) \downarrow,
\]

is r.e., but not recursive.

The details given in the following and Theorem 8.4 will be omitted in this lecture **Jump over details and Theorem 8.4**.

**Proof of Lemma 8.3**

(a) Assume \( A \subseteq \mathbb{N}^k \) is decidable.

(b) Assume \( B \subseteq \mathbb{N}^k \) is decidable.

Then

\[
\mathbb{N}^k \setminus A
\]

is recursive, therefore its characteristic function

\[
\chi_{\mathbb{N}^k \setminus A}
\]

is recursive as well.

Define

\[
f : \mathbb{N}^k \xrightarrow{\sim} \mathbb{N}, f(\vec{x}) :\equiv (\mu y. \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0).
\]

Note that \( y \) doesn’t occur in the body of the \( \mu \)-expression.

Then we have

If \( A(\vec{x}) \), then

\[
\chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0,
\]

so

\[
f(\vec{x}) \simeq (\mu y. \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0) \simeq 0,
\]

especially

\[
f(\vec{x}) \downarrow.
\]
**Proof of Lemma 8.3**

- If \((\mathbb{N}^k \setminus A)(\vec{x})\), then
  \[ \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 1 \]
  so there exists no \(y\) s.t.
  \[ \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0 \]
  therefore
  \[ f(\vec{x}) \simeq (\mu y. \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0) \simeq \bot \]
  especially
  \[ f(\vec{x}) \uparrow \]

**Proof of Lemma 8.3**

- So we get
  \[ A(\vec{x}) \iff f(\vec{x}) \downarrow \iff \vec{x} \in \text{dom}(f) \]
  \[ A = \text{dom}(f) \text{ is r.e.} \]

**Theorem 8.4**

There exist r.e. predicates
\[ W^n \subseteq \mathbb{N}^{n+1} \]
s.t., with
\[ W^n_e := \{ \vec{x} \in \mathbb{N}^n \mid W^n(e, \vec{x}) \} \]
we have the following:
- Each of the predicates \(W^n_e \subseteq \mathbb{N}^n\) is r.e.
- For each r.e. predicate \(P \subseteq \mathbb{N}^n\) there exists an \(e \in \mathbb{N}\) s.t. \(P = W^n_e\), i.e.
  \[ \forall \vec{x} \in \mathbb{N}. P(\vec{x}) \iff W^n_e(\vec{x}) \]

Theorem 8.4

Therefore, the r.e. sets $P \subseteq \mathbb{N}^n$ are exactly the sets $W^n_e$ for $e \in \mathbb{N}$.

Remark on Theorem 8.4

- $W^n_e$ is therefore a **universal recursively enumerable sets**, which encodes all other recursively enumerable sets.
- The theorem means that we can assign to every recursively enumerable predicate $A$ a natural number, namely the $e$ s.t. $A = W^n_e$.
- Each code denotes one predicate.
- However, several numbers denote the same predicate:
  - there are $e$, $e'$ s.t. $e \neq e'$, but $W^n_e = W^n_{e'}$.
  - (Since there are $e \neq e'$ s.t. $\{e\}^n = \{e'\}^n$).

Proof of Theorem 8.4

- Let $f_n$ s.t.
  \[ \forall e, \bar{n} \in \mathbb{N}. f_n(e, \bar{x}) \simeq \{e\}(\bar{x}) \]

Define

\[ W^n := \text{dom}(f_n) \]

$W^n$ is r.e.

We have

\[ \bar{x} \in W^n_e \iff (e, \bar{x}) \in W^n \]

\[ \iff f_n(e, \bar{x}) \downarrow \]

\[ \iff \{e\}(\bar{x}) \downarrow \]

\[ \iff \bar{x} \in \text{dom}(\{e\}^n) \]
Proof of Theorem 8.4

Therefore

\[ W_e^n = \text{dom}(\{e\}^n) . \]

\( W^n \) is r.e., since \( f_n \) is partial recursive.

Furthermore, we have for any set \( A \subseteq \mathbb{N}^n \):

\[
A \text{ is r.e.} \iff A = \text{dom}(f) \text{ for some partial recursive } f
\]

\[
A \text{ is r.e.} \iff A = \text{dom}(\{e\}^n) \text{ for some } e \in \mathbb{N}
\]

\[
A \text{ is r.e.} \iff A = W_e^n \text{ for some } e \in \mathbb{N}.
\]

This shows the assertion.

Theorem 8.5

Let \( A \subseteq \mathbb{N}^n \). The following is equivalent:

(i) \( A \) is r.e.

(ii) \( A = \{\vec{x} | \exists y. R(\vec{x}, y)\} \)

for some primitive recursive predicate \( R \).

(iii) \( A = \{\vec{x} | \exists y. R(\vec{x}, y)\} \)

for some recursive predicate \( R \).

(iv) \( A = \{\vec{x} | \exists y. R(\vec{x}, y)\} \)

for some recursively enumerable predicate \( R \).

Remark

We can summarise Theorem 8.5 as follows:

There are 3 equivalent ways of defining that \( A \subseteq \mathbb{N}^n \) is r.e.:

\( A = \text{dom}(f) \) for some partial recursive \( f \);

\( A = \emptyset \) or \( A \) is the image of primitive recursive/recursive functions \( f_0, \ldots, f_{n-1} \);

\( A = \{\vec{x} | \exists y. R(\vec{x}, y)\} \) for some primitive recursive/recursive/r.e. \( R \).
Remark, Case \(n = 1\)

- For \(A \subseteq \mathbb{N}\) the following is equivalent:
  - \(A\) is r.e.
  - \(A = \emptyset\) or \(A = \text{ran}(f)\) for some primitive recursive \(f : \mathbb{N} \to \mathbb{N}\).
  - \(A = \emptyset\) or \(A = \text{ran}(f)\) for some recursive \(f : \mathbb{N} \to \mathbb{N}\).

Therefore \(A \subseteq \mathbb{N}\) is r.e., if
  - \(A = \emptyset\)
  - or there exists a (prim.-)rec. function \(f\), which enumerates all its elements.

This explains the name “recursively enumerable predicate”.
Skip Proof.

---

Proof Idea for Theorem 8.5:

- ((i) \(\rightarrow\) (ii), Cont)
  - where
    - \(R(\bar{x}, y) \iff\) the TM for comp. \(f(\bar{x})\) termin. after \(y\) steps.
    - \(R\) is primitive recursive.

---

Proof Ideas

(ii) \(\rightarrow\) (v), special case \(n = 1\):

Assume
  - \(A = \{x \in \mathbb{N} \mid \exists y.R(x, y)\}\) where \(R\) is prim. rec.
  - \(A \neq \emptyset\),
  - \(y \in A\) fixed.

Define \(f : \mathbb{N} \to \mathbb{N}\) recursive,
  
  \[f(x) = \begin{cases} 
  \pi_0(x), & \text{if } R(\pi_0(x), \pi_1(x)) \\
  y, & \text{otherwise}.
  \end{cases} \]

Then \(A = \text{ran}(f)\).
Proof Ideas

(v), (vi) → (i), special case $n = 1$:
Assume

$$A = \text{ran}(f),$$
where $f$ is (prim.-)recursive.
Then

$$A = \text{dom}(g),$$
where

$$g(x) \simeq (\mu y. f(y) = x).$$
g is partial recursive.
The full details will be omitted in the lecture.
Skip Details

Proof of Theorem 8.5

(i) → (ii) (Cont.): (The actual predicate $R$ we will take will be slightly differently from that in the proof idea – it is technically easier to prove the theorem this way.)
If $A$ is r.e., then for some partial recursive function $f : N^n \rightarrow N$ we have

$$A = \text{dom}(f).$$

Let $f = \{e\}^n$.
By Kleene's Normal Form Theorem there exist a primitive recursive function $U : N \rightarrow N$ and a primitive recursive predicate $T_n \subseteq N^{n+1}$ s.t.

$$\{e\}^n(x) \simeq U(\mu y. T_n(e, x, y)).$$

Therefore

$$A(\vec{x}) \iff \vec{x} \in \text{dom}(f)$$
$$\iff \vec{x} \in \text{dom}(\{e\}^n)$$
$$\iff U(\mu y. T_n(e, \vec{x}, y)) \downarrow$$
U prim. rec., therefore total
$$\iff \mu y. T_n(e, \vec{x}, y) \downarrow$$
$$\iff \exists y. T_n(e, \vec{x}, y)$$
$$\iff \exists y. R(\vec{x}, y).$$

where

$$R(\vec{x}, y) \iff T_n(e, \vec{x}, y).$$

Proof of Theorem 8.5

(i) → (ii) (Cont.): Now $R$ is primitive recursive, and

$$A = \{\vec{x} \mid \exists y. R(\vec{x}, y)\}.$$
Proof of Theorem 8.5

(ii) → (iii): Trivial.

(iii) → (iv): By Lemma 8.3.

---

Proof of Theorem 8.5

(iv) → (ii), Cont.

Here

\[ R'(\vec{x}, y) \iff S(\vec{x}, \pi_0(y), \pi_1(y)) \] is primitive recursive.

---

Proof of Theorem 8.5

(ii) → (v):

Assume \( A \) is not empty and \( R \) is primitive recursive s.t.

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} \ . \]

Let \( \vec{z} = z_0, \ldots, z_{n-1} \) be some fixed elements s.t. \( A(\vec{z}) \) holds.

Define for \( i = 0, \ldots, n-1 \)

\[ f_i(x) := \begin{cases} \pi_i^{n+1}(x), & \text{if } R(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_n^{n+1}(x)) , \\ z_i, & \text{otherwise.} \end{cases} \]

\( f_i \) are primitive recursive.
Proof of Theorem 8.5

((ii) → (v), Cont.)

We show

\[ A = \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\} . \]

Proof of Theorem 8.5

((ii) → (v), Cont.)

("⊇", Cont.):

If \( (\mathbb{N}^k \setminus R)(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_n^{n+1}(x)) \), then

\[ f_i(x) = z_i , \]

therefore by \( A(\vec{z}) \)

\[ A(f_0(x), \ldots, f_{n-1}(x)) . \]

So in both cases we get that

\[ A(f_0(x), \ldots, f_{n-1}(x)) , \]

so

\[ \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\} \subseteq A . \]
Proof of Theorem 8.5

((ii) → (v), Cont.); (“⊆”, Cont)

Then we have
\[ x_i = \pi_i^{n+1}(z), \quad y = \pi_n^{n+1}(z), \]
therefore
\[ R(\pi_0^{n+1}(z), \pi_1^{n+1}(z), \ldots, \pi_{n-1}^{n+1}(z), \pi_n^{n+1}(z)) , \]
therefore for \( i = 0, \ldots, n-1 \)
\[ f_i(z) = \pi_i^{n+1}(z) = x_i , \]

Therefore we have shown
\[ A \subseteq \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\} , \]
and the assertion follows.
Proof of Theorem 8.5

((vi) → (i), Cont.)

Furthermore, we have

\[ A(x_0, \ldots, x_{n-1}) \iff \exists x \in \mathbb{N}. x_0 = f_0(x) \land \cdots \land x_{n-1} = f_{n-1}(x) \]

therefore

\[ A = \text{dom}(f) \text{ is r.e.} \]

---

Theorem 8.6

\( A \subseteq \mathbb{N}^k \) is recursive iff both \( A \) and \( \mathbb{N}^k \setminus A \) are r.e.

**Proof idea:**

“⇒” is easy.

For “⇐”:

Assume

\[ A(\bar{x}) \iff \exists y. R(\bar{x}, y) \]

\[ (\mathbb{N}^k \setminus A)(\bar{x}) \iff \exists y. S(\bar{x}, y) \]

In order to decide \( A \), search simultaneously for a \( y \) s.t. \( R(\bar{x}, y) \) and for a \( y \) s.t. \( S(\bar{x}, y) \) holds.

If we find a \( y \) s.t. \( R(\bar{x}, y) \) holds, then \( A(\bar{x}) \) holds.

If we find a \( y \) s.t. \( S(\bar{x}, y) \) holds, then \( \neg A(\bar{x}) \) holds.

The details of the proof will be omitted in this lecture.

---

Proof of Theorem 8.6, “⇒”

If \( A \) is recursive, then both \( A \) and \( \mathbb{N}^k \setminus A \) are recursive, therefore as well r.e.

---

Proof of Theorem 8.6, “⇐”

Assume \( A, \mathbb{N}^k \setminus A \) are r.e.

Then there exist primitive recursive predicates \( R \) and \( S \) s.t.

\[ A = \{ \bar{x} \mid \exists y. R(\bar{x}, y) \} \]

\[ \mathbb{N}^k \setminus A = \{ \bar{x} \mid \exists y. S(\bar{x}, y) \} \]

---

Jump over details
**Proof of Theorem 8.6, “⇐”**

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ N^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} . \]

- By
  \[ A \cup (N^k \setminus A) = N^k , \]
  it follows
  \[ \forall \vec{x}. ((\exists y. R(\vec{x}, y)) \lor (\exists y. S(\vec{x}, y))) , \]
  therefore as well
  \[ \forall \vec{x}. \exists y. (R(\vec{x}, y) \lor S(\vec{x}, y)) . \quad (*) \]

**Proof of Theorem 8.6, “⇐”**

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ N^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} , \]
\[ \forall \vec{x}. \exists y. (R(\vec{x}, y) \lor S(\vec{x}, y)) . \quad (*) \]

- Define
  \[ h : \mathbb{N}^n \to \mathbb{N} , \quad h(\vec{x}) := \mu y. (R(\vec{x}, y) \lor S(\vec{x}, y)) . \]

  - \( h \) is partial recursive.
  - By \((*)\) we have \( h \) is total, so \( h \) is recursive.
  - We show
    \[ A(\vec{x}) \iff R(\vec{x}, h(\vec{x})) . \]

**Proof of Theorem 8.6, “⇐”**

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ N^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} , \]
\[ h(\vec{x}) := \mu y. (R(\vec{x}, y) \lor S(\vec{x}, y)) , \]
\[ Show \ A(\vec{x}) \iff R(\vec{x}, h(\vec{x})) . \]

- If \( A(\vec{x}) \) then
  \[ \exists y. R(\vec{x}, y) \]
  and
  \[ \vec{x} \notin (N^k \setminus A) , \]
  therefore
  \[ \neg \exists y. S(\vec{x}, y) . \]

  Therefore we have for the \( y \) found by \( h(\vec{x}) \) that \( R(\vec{x}, y) \)
  holds, i.e.
  \[ R(\vec{x}, h(\vec{x})) . \]

**Proof of Theorem 8.6, “⇐”**

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ N^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} , \]
\[ h(\vec{x}) := \mu y. (R(\vec{x}, y) \lor S(\vec{x}, y)) , \]
\[ Show \ A(\vec{x}) \iff R(\vec{x}, h(\vec{x})) . \]

- On the other hand, if \( R(\vec{x}, h(\vec{x})) \) holds then
  \[ \exists y. R(\vec{x}, y) \]
  therefore
  \[ A(\vec{x}) . \]

  Therefore
  \[ A = \{ \vec{x} \mid R(\vec{x}, h(\vec{x})) \} \text{ is recursive.} \]
Theorem 8.7

Let \( f : \mathbb{N}^n \to \mathbb{N} \).
Then

\[ f \text{ is partial recursive } \iff G_f \text{ is r.e.} \]

Proof idea for \( \iff \):
Assume \( R \) primitive recursive s.t.

\[ G_f(\vec{x}, y) \iff \exists z. R(\vec{x}, y, z) \]

In order to compute \( f(\vec{x}) \), search for a \( y \) s.t. \( R(\vec{x}, \pi_0(y), \pi_1(y)) \) holds.
\( f(\vec{x}) \) will be the first projection of this \( y \).

The details of the proof will be omitted in this lecture.

Proof of Theorem 8.7, \( \Rightarrow \)

Assume \( f \) is partial recursive.
Then \( f = \{e\}^n \) for some \( e \in \mathbb{N} \).
By Kleene’s Normal Form Theorem we have

\[ f(\vec{x}) \simeq U(\mu y. T_n(\vec{x}, y)) \]

for some primitive recursive relation

\[ T_n \subseteq \mathbb{N}^{n+1} \]

and some primitive recursive function

\[ U : \mathbb{N} \to \mathbb{N} \]

Proof of Theorem 8.7, \( \Leftarrow \)

If \( G_f \) is r.e., then there exists a primitive recursive predicate \( R \) s.t.

\[ f(\vec{x}) \simeq y \iff (\vec{x}, y) \in G_f \iff \exists z. R(\vec{x}, y, z) \]

Therefore for any \( z \) s.t. \( R(\vec{x}, \pi_0(z), \pi_1(z)) \) holds we have that

\[ f(\vec{x}) \simeq \pi_0(z) \]

Therefore

\[ f(\vec{x}) \simeq \pi_0(\mu u. R(\vec{x}, \pi_0(u), \pi_1(u))) \]

\( f \) is partial recursive.
Lemma 8.8

The recursively enumerable sets are closed under:

(a) **Union:**
If \( A, B \subseteq \mathbb{N}^n \) are r.e., so is \( A \cup B \).

(b) **Intersection:**
If \( A, B \subseteq \mathbb{N}^n \) are r.e., so is \( A \cap B \).

(c) **Substitution by recursive functions:**
If \( A \subseteq \mathbb{N}^n \) is r.e., \( f_i : \mathbb{N}^k \rightarrow \mathbb{N} \) are recursive for \( i = 0, \ldots, n \), so is

\[
C := \{ \vec{y} \in \mathbb{N}^k \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y})) \} .
\]

(d) **(Unbounded) existential quantification:**
If \( D \subseteq \mathbb{N}^{n+1} \) is r.e., so is

\[
E := \{ x \in \mathbb{N}^n \mid \exists y. D(x, y) \} .
\]

(e) **Bounded universal quantification:**
If \( D \subseteq \mathbb{N}^{n+1} \) is r.e., so is

\[
F := \{ (x, z) \in \mathbb{N}^{n+1} \mid \forall y < z. D(x, z) \} .
\]

The details of the proof will be omitted in this lecture.

Jump over details

Proof of Lemma 8.8

Let \( A, B \subseteq \mathbb{N}^n \) be r.e.

Then there exist primitive recursive relations \( R, S \) s.t.

\[
A = \{ x \in \mathbb{N}^n \mid \exists y. R(x, y) \} , \\
B = \{ x \in \mathbb{N}^n \mid \exists y. S(x, y) \} .
\]

One can easily see that

\[
A \cup B = \{ x \in \mathbb{N}^n \mid \exists y. (R(x, y) \lor S(x, y)) \} , \\
A \cap B = \{ x \in \mathbb{N}^n \mid \exists y. (R(x, \pi_0(y)) \land S(x, \pi_1(y))) \} .
\]

therefore \( A \cup B \) and \( A \cap B \) are r.e.
Proof of Lemma 8.8 (c)

\[ A = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. R(\vec{x}, y) \} , \]
\[ B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. S(\vec{x}, y) \} . \]

Assume \( A \subseteq \mathbb{N}^n \) is r.e., \( f_i : \mathbb{N}^k \to \mathbb{N} \) are recursive for \( i = 0, \ldots, n \).

Need to show that
\[ C := \{ (\vec{y} \in \mathbb{N}^k \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y})) \} . \]
is r.e.

Follows by
\[ C = \{ \vec{y} \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y})) \} = \{ \vec{y} \mid \exists z. R(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y}), z) \} \] is r.e.

Proof of Lemma 8.8 (d), (e)

(d) follows from Theorem 8.5.

(e):

Assume \( T \) is a primitive recursive predicate s.t.
\[ D = \{ (\vec{x}, y) \in \mathbb{N}^{n+1} \mid \exists z. T(\vec{x}, y, z) \} . \]

Then we get
\[ F = \{ (\vec{x}, y) \mid \forall y' < y.D(\vec{x}, y') \} = \{ (\vec{x}, y) \mid \forall y' < y. \exists z.T(\vec{x}, y', z) \} = \{ (\vec{x}, y) \mid \exists z. \forall y' < y.T(\vec{x}, y', (z)_{y'}) \} \] is r.e.,

where in the last line we used that
\[ \{ (\vec{x}, z) \mid \forall y' < y.T(\vec{x}, y', (z)_{y'}) \} \] is primitive recursive.

Lemma 8.9

The r.e. predicates are not closed under complement:
There exists an r.e. predicate \( A \subseteq \mathbb{N}^n \) s.t. \( \mathbb{N}^n \setminus A \) is not r.e.

Proof:

- \( \text{Halt}^n \) is r.e.
- \( \mathbb{N}^n \setminus \text{Halt}^n \) is not r.e.
  - Otherwise by Theorem 8.6 \( \text{Halt}^n \) would be recursive.
  - But by Lemma 8.3 (b) \( \text{Halt}^n \) is not recursive.