Sec. 8: Semi-Computable Predicates

- We study \( P \subseteq \mathbb{N}^n \), which are
  - not decidable,
  - but “half decidable”.
- Official name is
  - semi-decidable,
  - or semi-computable.
  - or recursively enumerable (r.e.).

Rec. enum. vs. semi-decidable

- **Recursively enumerable** stands for the definition based on the notion of partial recursive functions.
- **Semi-decidable** or semi-computable stand for the definition based on an intuitive notion of “(partial) computable function”
- Assuming the **Church-Turing thesis**, the two notions coincide.

Rec. Sets

- Remember:
  - A predicate \( A \) is recursive, iff \( \chi_A \) is recursive.
  - So we have a “full” decision procedure:
    \[
    P(\vec{x}) \iff \chi_P(\vec{x}) = 1, \text{ i.e. answer yes } ,
    \]
    \[
    \neg P(\vec{x}) \iff \chi_P(\vec{x}) = 0, \text{ i.e. answer no } .
    \]

Semi-Decidable Sets

- \( P \subseteq \mathbb{N}^n \) will be semi-decidable,
  if there exists a partial recursive recursive function \( f \) s.t.
  \[
  P(\vec{x}) \iff f(\vec{x}) \downarrow .
  \]
- If \( P(\vec{x}) \) holds,
  we will eventually know it:
  the algorithm for computing \( f \) will finally terminate,
  and then we know that \( P(\vec{x}) \) holds.
- If \( P(\vec{x}) \) doesn’t hold,
  then the algorithm computing \( f \) will loop for ever,
  and we never get an answer.
Semi-Decidable Sets

So we have:

\[ P(\vec{x}) \iff f(\vec{x}) \downarrow \text{ i.e. answer yes} , \]

\[ \neg P(\vec{x}) \iff f(\vec{x}) \uparrow \text{ i.e. no answer returned by } f . \]

Applications

One might think that semi-computable sets don’t occur in computing.

But they occur in many applications.

 Examples are

- Checking whether a program terminates is semi-decidable.
- Checking whether a program in C++ is type correct is because of the template mechanism semi-decidable.
- In C++ compilers this problem is usually prevented by having a flag which limits the number of times templates are unfolded.

Examples (Cont.)

- Type checking in Agda (used in the module interactive theorem proving) is semi-decidable.
- Does in most applications not cause any problems.

Jump over next example

Whether a statement is provable in many logical systems is semi-decidable.

But even so this is semi-decidable, many search algorithms succeed in most practical cases.

Often one can predict a certain time, after which normally the search algorithm should have returned an answer.

- If the search algorithm hasn’t returned an answer after this time it is likely (but not guaranteed) that the statement is unprovable.
Def. 8.1 (Recursively Enumerable)

A predicate \( A \subseteq \mathbb{N}^n \) is \textit{recursively enumerable}, in short \textit{r.e.}, if there exists a partial recursive function \( f : \mathbb{N}^n \xrightarrow{*} \mathbb{N} \) s.t.

\[
A = \text{dom}(f).
\]

Sometimes recursive predicates are as well called
- \textit{semi-decidable}
- \textit{semi-computable}
- \textit{partially computable}.

Lemma 8.3

(a) Every recursive predicate is r.e.
(b) The \textit{halting problem}, i.e.

\[
\text{Halt}^n(e, \vec{x}) : \iff \{e\}^n(\vec{x}) \downarrow,
\]

is r.e., but not recursive.

The proof of Lemma 8.3 and the statement and proof of Theorem 8.4 will be omitted in this lecture.

Proof of Lemma 8.3

(a) Assume \( A \subseteq \mathbb{N}^k \) is decidable.
   Then
   \[
   \mathbb{N}^k \setminus A
   \]
   is recursive, therefore its characteristic function
   \[
   \chi_{\mathbb{N}^k \setminus A}
   \]
   is recursive as well.
   Define
   \[
   f : \mathbb{N}^k \xrightarrow{*} \mathbb{N}, f(\vec{x}) : \equiv (\mu y. \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0).
   \]
   Note that \( y \) doesn’t occur in the body of the \( \mu \)-expression.

Proof of Lemma 8.3

Then we have
   If \( A(\vec{x}) \), then
   \[
   \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0,
   \]
   so
   \[
   f(\vec{x}) \simeq (\mu y. \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0) \simeq 0,
   \]
   especially
   \[
   f(\vec{x}) \downarrow.
   \]
Proof of Lemma 8.3

If \((\mathbb{N}^k \setminus A)(\vec{x})\), then
\[
\chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 1,
\]
so there exists no \(y\) s.t.
\[
\chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0.
\]
therefore
\[
f(\vec{x}) \simeq (\mu y. \chi_{\mathbb{N}^k \setminus A}(\vec{x}) \simeq 0) \simeq \bot,
\]
especially
\[
f(\vec{x}) \uparrow.
\]

Proof of Lemma 8.3

So we get
\[
A(\vec{x}) \iff f(\vec{x}) \downarrow \iff \vec{x} \in \text{dom}(f),
\]

A = \text{dom}(f) is r.e.

(b) We have
\[
\text{Halt}^n(e, \vec{x}) :\iff f_n(e, \vec{x}) \downarrow,
\]
where \(f_n\) is partial recursive as in Sect. 5 s.t.
\[
\{e\}^n(\vec{x}) \simeq f_n(e, \vec{x}).
\]
So
\[
\text{Halt}^n = \text{dom}(f_n) \text{ is r.e.}
\]
We have seen above that Halt\(^n\) is non-computable, i.e. not recursive.

Jump over Theorem 8.4.

Theorem 8.4

There exist r.e. predicates
\[
W^n \subseteq \mathbb{N}^{n+1}
\]
s.t., with
\[
W^n_e := \{ \vec{x} \in \mathbb{N}^n \mid W^n(e, \vec{x}) \},
\]
we have the following:

- Each of the predicates \(W^n_e \subseteq \mathbb{N}^n\) is r.e.
- For each r.e. predicate \(P \subseteq \mathbb{N}^n\) there exists an \(e \in \mathbb{N}\) s.t. \(P = W^n_e\), i.e.

\[
\forall \vec{x} \in \mathbb{N}. P(\vec{x}) \iff W^n_e(\vec{x}).
\]
Theorem 8.4

Therefore, the r.e. sets $P \subseteq \mathbb{N}^n$ are exactly the sets $W^n_e$ for $e \in \mathbb{N}$.

Remark on Theorem 8.4

- $W^n_e$ is therefore a **universal recursively enumerable sets**, which encodes all other recursively enumerable sets.

- The theorem means that that we can assign to every recursively enumerable predicate $A$ a natural number, namely the $e$ s.t. $A = W^n_e$.

- Each code denotes one predicate.

- However, several numbers denote the same predicate:
  - there are $e, e'$ s.t. $e \neq e'$, but $W^n_e = W^n_{e'}$.
  - (Since there are $e \neq e'$ s.t. $\{e\}^n = \{e'\}^n$).

Proof Idea for Theorem 8.4

$W^n_e := \text{dom}(\{e\}^n)$.

If $A$ is r.e., then $A = \text{dom}(f)$ for some partial rec. $f$.

Let $f = \{e\}^n$.

Then $A = W^n_e$.

The details given in the following will be omitted in the lecture. **Jump over Details**

Proof of Theorem 8.4

- Let $f_n$ s.t.
  \[ \forall e, \vec{n} \in \mathbb{N}. f_n(e, \vec{x}) \simeq \{e\}(\vec{x}). \]

- Define
  \[ W^n := \text{dom}(f_n). \]

- $W^n$ is r.e.

- We have
  \[ \vec{x} \in W^n_e \iff (e, \vec{x}) \in W^n \]
  \[ \iff f_n(e, \vec{x}) \downarrow \]
  \[ \iff \{e\}(\vec{x}) \downarrow \]
  \[ \iff \vec{x} \in \text{dom}(\{e\}^n). \]
Proof of Theorem 8.4

Therefore
\[ W_e^n = \text{dom}(\{e\}^n) \, . \]

\( W_n \) is r.e., since \( f_n \) is partial recursive.

Furthermore, we have for any set \( A \subseteq \mathbb{N}^n \)

\[ A \text{ is r.e. } \iff A = \text{dom}(f) \text{ for some partial recursive } f \]
\[ \iff A = \text{dom}(\{e\}^n) \text{ for some } e \in \mathbb{N} \]
\[ \iff A = W_e^n \text{ for some } e \in \mathbb{N} \, . \]

This shows the assertion.

Theorem 8.5

Let \( A \subseteq \mathbb{N}^n \). The following is equivalent:

(i) \( A \) is r.e.

(ii) \( A = \{\vec{x} \mid \exists y. R(\vec{x}, y)\} \)

for some primitive recursive predicate \( R \).

(iii) \( A = \{\vec{x} \mid \exists y. R(\vec{x}, y)\} \)

for some recursive predicate \( R \).

(iv) \( A = \{\vec{x} \mid \exists y. R(\vec{x}, y)\} \)

for some recursively enumerable predicate \( R \).

Remark

We can summarise Theorem 8.5 as follows: There are 3 equivalent ways of defining that \( A \subseteq \mathbb{N}^n \) is r.e.:

A = \text{dom}(f) \text{ for some partial recursive } f;

A = \emptyset \text{ or } A \text{ is the image of primitive recursive/recursive functions } f_0, \ldots, f_{n-1};

A = \{\vec{x} \mid \exists y. R(\vec{x}, y)\} \text{ for some primitive recursive/recursive/r.e. } R.
Remark, Case $n = 1$

- For $A \subseteq \mathbb{N}$ the following is equivalent:
  - $A$ is r.e.
  - $A = \emptyset$ or $A = \operatorname{ran}(f)$ for some primitive recursive $f : \mathbb{N} \to \mathbb{N}$.
  - $A = \emptyset$ or $A = \operatorname{ran}(f)$ for some recursive $f : \mathbb{N} \to \mathbb{N}$.

- Therefore $A \subseteq \mathbb{N}$ is r.e., if
  - $A = \emptyset$
  - or there exists a (prim.-)rec. function $f$, which enumerates all its elements.

- This explains the name “recursively enumerable predicate”.
  Skip Proof.

Proof Ideas

- (ii) $\implies$ (v), special case $n = 1$:
  Assume
  - $A = \{x \in \mathbb{N} \mid \exists y.R(x, y)\}$ where $R$ is prim. rec.
  - $A \neq \emptyset$,
  - $y \in A$ fixed.

Define $f : \mathbb{N} \to \mathbb{N}$ recursive,

$$f(x) = \begin{cases} \pi_0(x), & \text{if } R(\pi_0(x), \pi_1(x)), \\ y, & \text{otherwise}. \end{cases}$$

Then $A = \operatorname{ran}(f)$.
Proof of Theorem 8.5

(i) → (ii) (Cont.)

Therefore

\[ A(\bar{x}) \iff \bar{x} \in \text{dom}(f) \]
\[ \iff \bar{x} \in \text{dom}(\{e\}^n) \]
\[ \iff U(\mu y. T_n(e, \bar{x}, y)) \downarrow \]

U prim. rec., therefore total
\[ \iff \mu y. T_n(e, \bar{x}, y) \downarrow \]
\[ \iff \exists y. T_n(e, \bar{x}, y) \]
\[ \iff \exists y. R(\bar{x}, y). \]

where

\[ R(\bar{x}, y) \iff T_n(e, \bar{x}, y). \]

Proof of Theorem 8.5

(i) → (ii):

(The actual predicate \( R \) we will take will be slightly differently from that in the proof idea – it is technically easier to prove the theorem this way.)

If \( A \) is r.e., then for some partial recursive function \( f : \mathbb{N}^n \rightarrow \mathbb{N} \) we have

\[ A = \text{dom}(f). \]

Let \( f = \{e\}^n \).

By Kleene’s Normal Form Theorem there exist a primitive recursive function \( U : \mathbb{N} \rightarrow \mathbb{N} \) and a primitive recursive predicate \( T_n \subseteq \mathbb{N}^{n+1} \) s.t.

\[ \{e\}^n(\bar{x}) \simeq U(\mu y. T_n(e, \bar{x}, y)). \]
Proof of Theorem 8.5

(ii) → (iii): Trivial.

(iii) → (iv): By Lemma 8.3.

(iv) → (ii), Cont.

Here

\[ R'(\vec{x}, y) :\Leftrightarrow S(\vec{x}, \pi_0(y), \pi_1(y)) \]

is primitive recursive.

Proof of Theorem 8.5

(ii) → (v):

Assume \( A \) is not empty and \( R \) is primitive recursive s.t.

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} . \]

Let \( \vec{z} = z_0, \ldots, z_{n-1} \) be some fixed elements s.t. \( A(\vec{z}) \) holds.

Define for \( i = 0, \ldots, n-1 \)

\[ f_i(x) := \begin{cases} \pi_i^{n+1}(x), & \text{if } R(\pi_0^{n+1}(x), \pi_1^{n+1}(x), \ldots, \pi_{n-1}^{n+1}(x), \pi_n^{n+1}(x)), \\ z_i, & \text{otherwise}. \end{cases} \]

\( f_i \) are primitive recursive.
Proof of Theorem 8.5

((ii) → (v), Cont.)

We show

\[ A = \{ (f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N} \} . \]

((ii) → (v), Cont.)

("\supseteq", Cont.):

Assume \( x \in \mathbb{N} \), and show

\[ \exists \, z. (f_0(z) = x_0 \land \cdots \land f_{n-1}(z) = x_{n-1}) . \]

We have for some \( y \)

\[ R(x_0, \ldots, x_{n-1}, y) . \]

Let

\[ z = \pi^{n+1}(x_0, \ldots, x_{n-1}, y) . \]
Proof of Theorem 8.5

(ii) → (v), (Cont.); (“⊆”, Cont)

Then we have
\[ x_i = \pi_i^{n+1}(z), \quad y = \pi_n^{n+1}(z), \]

therefore
\[ R(\pi_0^{n+1}(z), \pi_1^{n+1}(z), \ldots, \pi_n^{n+1}(z), \pi_n^{n+1}(z)), \]

therefore for \( i = 0, \ldots, n-1 \)
\[ f_i(z) = \pi_i^{n+1}(z) = x_i, \]

Proof of Theorem 8.5

(vi) → (i), (Cont.)

\( f \) can be written as
\[ f(x_0, \ldots, x_{n-1}) \simeq \mu x. (((f_0(x) \div x_0) + (x_0 \div f_0(x)) + ((f_1(x) \div x_1) + (x_1 \div f_1(x)) + \cdots + ((f_{n-1}(x) \div x_{n-1}) + (x_{n-1} \div f_{n-1}(x))) \simeq 0), \]

therefore \( f \) is partial recursive.

Proof of Theorem 8.5

(v) → (vi): Trivial.

(vi) → (i):

If \( A \) is empty, then \( A \) is recursive, therefore r.e.

Assume
\[ A = \{(f_0(x), \ldots, f_{n-1}(x)) \mid x \in \mathbb{N}\} . \]

for some recursive functions \( f_i \).

Define
\[ f : \mathbb{N}^n \rightarrow \mathbb{N} , \]

s.t.
\[ f(x_0, \ldots, x_{n-1}) \simeq \mu x. (f_0(x) \simeq x_0 \land \cdots \land f_{n-1}(x) \simeq x_{n-1}) . \]
Proof of Theorem 8.5

((vi) → (i), Cont.)

Furthermore, we have

\[ A(x_0, \ldots, x_{n-1}) \iff \exists x \in \mathbb{N} . x_0 = f_0(x) \land \cdots \land x_{n-1} = f_{n-1}(x) \]

therefore

\[ A = \text{dom}(f) \text{ is r.e.} . \]

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Theorem 8.6

\[ A \subseteq \mathbb{N}^k \text{ is recursive iff both } A \text{ and } \mathbb{N}^k \setminus A \text{ are r.e.} \]

**Proof idea:**

“⇒” is easy.

For “⇐”: Assume

\[
A(\vec{x}) \iff \exists y. R(\vec{x}, y) \\
(\mathbb{N}^k \setminus A)(\vec{x}) \iff \exists y. S(\vec{x}, y)
\]

In order to decide \( A \), search simultaneously for a \( y \) s.t. \( R(\vec{x}, y) \) and for a \( y \) s.t. \( S(\vec{x}, y) \) holds.

If we find a \( y \) s.t. \( R(\vec{x}, y) \) holds, then \( A(\vec{x}) \) holds.

If we find a \( y \) s.t. \( S(\vec{x}, y) \) holds, then \( \neg A(\vec{x}) \) holds.

The details of the proof will be omitted in this lecture.

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Proof of Theorem 8.6, “⇒”

If \( A \) is recursive, then both \( A \) and \( \mathbb{N}^k \setminus A \) are recursive, therefore as well r.e.

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Proof of Theorem 8.6, “⇐”

Assume \( A, \mathbb{N}^k \setminus A \) are r.e.

Then there exist primitive recursive predicates \( R \) and \( S \) s.t.

\[
A = \{ \vec{x} | \exists y. R(\vec{x}, y) \} , \\
\mathbb{N}^k \setminus A = \{ \vec{x} | \exists y. S(\vec{x}, y) \}.
\]
Proof of Theorem 8.6, “⇐”

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ N^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} . \]

- By
  \[ A \cup (N^k \setminus A) = N^k , \]
  it follows
  \[ \forall \vec{x}. ((\exists y. R(\vec{x}, y)) \lor (\exists y. S(\vec{x}, y))) , \]
  therefore as well
  \[ \forall \vec{x}. \exists y. (R(\vec{x}, y) \lor S(\vec{x}, y)) . \quad (\ast) \]

\[ CS_{226} \text{ Computability Theory, Michaelmas Term 2008, Sect. 8} \]

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Proof of Theorem 8.6, “⇐”

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ N^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} , \]
\[ h(\vec{x}) := \mu y. (R(\vec{x}, y) \lor S(\vec{x}, y)) , \]
Show \( A(\vec{x}) \leftrightarrow R(\vec{x}, h(\vec{x})) . \)

- If \( A(\vec{x}) \) then
  \[ \exists y. R(\vec{x}, y) \]
  and
  \[ \vec{x} \notin (N^k \setminus A) , \]
  therefore
  \[ \neg \exists y. S(\vec{x}, y) . \]
  Therefore we have for the \( y \) found by \( h(\vec{x}) \) that \( R(\vec{x}, y) \)
  holds, i.e.
  \[ R(\vec{x}, h(\vec{x})) . \]

\[ CS_{226} \text{ Computability Theory, Michaelmas Term 2008, Sect. 8} \]

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Proof of Theorem 8.6, “⇐”

\[ A = \{ \vec{x} \mid \exists y. R(\vec{x}, y) \} , \]
\[ N^k \setminus A = \{ \vec{x} \mid \exists y. S(\vec{x}, y) \} , \]
\[ h(\vec{x}) := \mu y. (R(\vec{x}, y) \lor S(\vec{x}, y)) , \]
Show \( A(\vec{x}) \leftrightarrow R(\vec{x}, h(\vec{x})) . \)

- On the other hand, if \( R(\vec{x}, h(\vec{x})) \) holds then
  \[ \exists y. R(\vec{x}, y) \]
  therefore
  \[ A(\vec{x}) . \]
  Therefore
  \[ A = \{ \vec{x} \mid R(\vec{x}, h(\vec{x})) \} \text{ is recursive.} \]
**Theorem 8.7**

Let \( f : \mathbb{N}^n \rightarrow \mathbb{N} \).

Then

\[ f \text{ is partial recursive } \iff G_f \text{ is r.e.} \]

**Proof idea for “\(\Leftarrow\)”**

Assume \( R \) primitive recursive s.t.

\[ G_f(x, y) \iff \exists z. R(x, y, z) \]

In order to compute \( f(x) \), search for a \( y \) s.t. \( R(x, \pi_0(y), \pi_1(y)) \) holds.

\( f(x) \) will be the first projection of this \( y \).

The details of the proof will be omitted in this lecture.

**Proof of Theorem 8.7, “\(\Rightarrow\)”**

Assume \( f \) is partial recursive.

Then \( f = \{e\}^n \) for some \( e \in \mathbb{N} \).

By Kleene’s Normal Form Theorem we have

\[ f(x) \simeq U(\mu y. T_n(x, y)) \]

for some primitive recursive relation

\[ T_n \subseteq \mathbb{N}^{n+1} \]

and some primitive recursive function

\[ U : \mathbb{N} \rightarrow \mathbb{N} \]

**Proof of Theorem 8.7, “\(\Leftarrow\)”**

If \( G_f \) is r.e., then there exists a primitive recursive predicate \( R \) s.t.

\[ f(x) \simeq y \iff (x, y) \in G_f \iff \exists z. R(x, y, z) \]

Therefore for any \( z \) s.t. \( R(x, \pi_0(z), \pi_1(z)) \) holds we have that

\[ f(x) \simeq \pi_0(z) \]

Therefore

\[ f(x) \simeq \pi_0(\mu u. R(x, \pi_0(u), \pi_1(u))) \]

\( f \) is partial recursive.
Lemma 8.8

The recursively enumerable sets are closed under:

(a) **Union** (and therefore $\lor$):
If $A, B \subseteq \mathbb{N}^n$ are r.e., so is $A \cup B$.

(b) **Intersection** (and therefore $\land$):
If $A, B \subseteq \mathbb{N}^n$ are r.e., so is $A \cap B$.

(c) **Substitution by recursive functions:**
If $A \subseteq \mathbb{N}^n$ is r.e., $f_i : \mathbb{N}^k \to \mathbb{N}$ are recursive for $i = 0, \ldots, n$, so is
$$C := \{ \vec{y} \in \mathbb{N}^k \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y})) \}.$$ 

(d) **(Unbounded) existential quantification:**
If $D \subseteq \mathbb{N}^{n+1}$ is r.e., so is
$$E := \{ \vec{x} \in \mathbb{N}^n \mid \exists y. D(\vec{x}, y) \}.$$ 

(e) **Bounded universal quantification:**
If $D \subseteq \mathbb{N}^{n+1}$ is r.e., so is
$$F := \{ (\vec{x}, z) \in \mathbb{N}^{n+1} \mid \forall y < z. D(\vec{x}, z) \}.$$ 

The details of the proof will be omitted in this lecture. Jump over details.

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Proof of Lemma 8.8

Let $A, B \subseteq \mathbb{N}^n$ be r.e.

Then there exist primitive recursive relations $R, S$ s.t.
$$A = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. R(\vec{x}, y) \} ,
B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. S(\vec{x}, y) \} .$$

One can easily see that
$$A \cup B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. (R(\vec{x}, y) \lor S(\vec{x}, y)) \} ,$$
$$A \cap B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. (R(\vec{x}, \pi_0(y)) \land S(\vec{x}, \pi_1(y))) \} .$$

Therefore $A \cup B$ and $A \cap B$ are r.e.
Proof of Lemma 8.8 (c)

\[ A = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. R(\vec{x}, y) \} , \]
\[ B = \{ \vec{x} \in \mathbb{N}^n \mid \exists y. S(\vec{x}, y) \} . \]

Assume \( A \subseteq \mathbb{N}^n \) is r.e., \( f_i : \mathbb{N}^k \to \mathbb{N} \) are recursive for \( i = 0, \ldots, n \).

Need to show that
\[ C := \{ (\vec{y} \in \mathbb{N}^k \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y})) \} \]
is r.e.
Follows by
\[ C \quad = \quad \{ \vec{y} \mid A(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y})) \} \]
\[ \quad = \quad \{ \vec{y} \mid \exists z. R(f_0(\vec{y}), \ldots, f_{n-1}(\vec{y}), z) \} \]
is r.e.

Proof of Lemma 8.8 (d), (e)

(d) follows from Theorem 8.5.
(e):

Assume \( T \) is a primitive recursive predicate s.t.
\[ D = \{ (\vec{x}, y) \in \mathbb{N}^{n+1} \mid \exists z. T(\vec{x}, y, z) \} . \]

Then we get
\[ F = \{ (\vec{x}, y) \mid \forall y' < y. D(\vec{x}, y') \} \]
\[ = \{ (\vec{x}, y) \mid \forall y' < y. \exists z. T(\vec{x}, y', z) \} \]
\[ = \{ (\vec{x}, y) \mid \exists z. \forall y'. y' < y. T(\vec{x}, y', (z)_{y'}) \} \]
is r.e.,

where in the last line we used that
\[ \{ (\vec{x}, z) \mid \forall y'. y' < y. T(\vec{x}, y', (z)_{y'}) \} \]
is primitive recursive.

Lemma 8.9

The r.e. predicates are not closed under complement:
There exists an r.e. predicate \( A \subseteq \mathbb{N}^n \) s.t. \( \mathbb{N}^n \setminus A \) is not r.e.

Proof:

• \( \text{Halt}^n \) is r.e.
• \( \mathbb{N}^n \setminus \text{Halt}^n \) is not r.e.
  • Otherwise by Theorem 8.6 \( \text{Halt}^n \) would be recursive.
  • But by Lemma 8.3. (b) \( \text{Halt}^n \) is not recursive.