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B3. Data Types

(a) The set of Booleans.
(b) The finite sets.
(c) Atomic formulae and the traffic light example.
(d) The disjoint union of sets.
(e) The Σ-set.
(f) The set of natural numbers.
(g) Lists.
(h) Universes.
(i) Algebraic data types.

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(a) The Set of Booleans

Formation Rule

$$\mathrm{Bool} : \mathrm{Set}$$

Introduction Rules

$$\mathrm{tt} : \mathrm{Bool} \qquad\qquad \mathrm{ff} : \mathrm{Bool}$$

Elimination Rule

$$\frac{C : \mathrm{Bool} \to \mathrm{Set} \quad ic : C\ \mathrm{tt} \quad ec : C\ \mathrm{ff} \quad cond}{\mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec\ cond : C\ cond}$$

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The Set of Booleans (Cont.)

Equality Rules

$$\frac{C : \mathrm{Bool} \to \mathrm{Set} \quad ic : C\ \mathrm{tt} \quad ec : C\ \mathrm{ff}}{\mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec\ \mathrm{tt} = ic : C\ \mathrm{tt}}$$

$$\frac{C : \mathrm{Bool} \to \mathrm{Set} \quad ic : C\ \mathrm{tt} \quad ec : C\ \mathrm{ff}}{\mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec\ \mathrm{ff} = ec : C\ \mathrm{ff}}$$

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Remarks

- In the above
  - tt stands for true, ff stands for false.
  - ic stands for "if-case", ec for "else-case".
  - con for "condition".
- We can write the elimination rule in a **more compact** but

  - $$\begin{aligned}\text{Case}_{\text{Bool}} : (C : \text{Bool} \to \text{Set}) \\ \to (ic : C\ \text{tt}) \\ \to (ec : C\ \text{ff}) \\ \to (cond : \text{Bool}) \\ \to C\ cond\end{aligned}$$.

- tt, ff are the **constructors** of Bool.

Remarks (Cont.)

- Notice that we then get for $C : \mathrm{Bool} \to \mathrm{Set}$, $ic : C\ \mathrm{tt}, ec$
  - $f := \mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec$,
    $: (cond : \mathrm{Bool}) \to C\ cond$
  - $f\ \mathrm{tt} = \mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec\ \mathrm{tt} = ic : C\ \mathrm{tt}$,
  - $f\ \mathrm{ff} = \mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec\ \mathrm{ff} = ec : C\ \mathrm{ff}$.
- So we obtain functions from Bool into other sets **withou**
  $\lambda(\mathbf{b} : \mathbf{Bool}). \cdots$.
- That's why we choose the argument to eliminate from as

Remarks (Cont.)

- This is similar to the definition of for instance $(+)$ in **curri**
  - $(+) : \mathrm{int} \rightarrow \mathrm{int} \rightarrow \mathrm{int}$.
  - $(+)\ 3$ is the function which takes an integer and adds t
    - **Shorter** than writing $\lambda x.3 + x$.

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Remarks (Cont.)

- Note that we have the following **order of the arguments**
  - First we have the **set into which we eliminate**.
  - Then follow the **cases**, one for each constructor.
  - Finally we put the **element which we are eliminating**
- In some sense $\mathrm{Case}_{\mathrm{Bool}}$ is a "$\mathrm{then}\ _\mathrm{else}\ _\mathrm{if}$ " – the **condi**
  last one.

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Example

$$\begin{array}{rcl} \textbf{AND} & := & \lambda(b, c : \mathrm{Bool}).\mathrm{Case}_{\mathrm{Bool}}\ (\lambda(b' : \mathrm{Bool}).\mathrm{B} \\ & : & \mathrm{Bool} \to \mathrm{Bool} \to \mathrm{Bool} \end{array}$$

- AND is the **conjunction**:
  - $\mathrm{AND}\ \mathrm{tt}\ c = c$.
    Correct since $\mathrm{tt} \wedge c = c$.
  - $\mathrm{AND}\ \mathrm{ff}\ c = \mathrm{ff}$.
    Correct since $\mathrm{ff} \wedge c = \mathrm{ff}$.

- In the following we write Bool, if it
  - is a type in **boldface red**,
  - and if it is a term, in *italic blue*.

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Example (Cont.)

- Derivation of $\mathrm{AND} : \mathbf{Bool} \to \mathbf{Bool} \to \mathbf{Bool}$:
  - First we derive $b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \lambda(b' : \mathbf{Bool}).\mathit{Bool}$

$$\mathit{Bool} : \mathrm{Set}$$
$$\mathbf{Bool} : \mathrm{Type}$$
$$b : \mathbf{Bool} \Rightarrow \mathrm{Context}$$
$$b : \mathbf{Bool} \Rightarrow \mathit{Bool} : \mathrm{Set}$$
$$b : \mathbf{Bool} \Rightarrow \mathbf{Bool} : \mathrm{Type}$$
$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{Context}$$
$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathit{Bool} : \mathrm{Set}$$
$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathbf{Bool} : \mathrm{Type}$$
$$b : \mathbf{Bool}, c : \mathbf{Bool}, b' : \mathbf{Bool} \Rightarrow \mathrm{Conte}$$
$$b : \mathbf{Bool}, c : \mathbf{Bool}, b' : \mathbf{Bool} \Rightarrow \mathit{Bool} :$$
$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \lambda(b' : \mathbf{Bool}).\mathit{Bool} : \mathbf{Bo}$$

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Example (Cont.)

- We derive

$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathbf{Bool} = (\lambda(b' : \mathbf{Bool}).\mathit{Bool}$$

(using part of the derivation above):

$$\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{b : \mathbf{Bool}, c : \mathbf{Bool}, b' : \mathbf{Bool} \Rightarrow \mathrm{Context}}}{b : \mathbf{Bool}, c : \mathbf{Bool}, b' : \mathbf{Bool} \Rightarrow \mathit{Bool} : \mathrm{Set}} \quad \dfrac{b : \mathbf{Bool}, c :}{b : \mathbf{Bool}, c :}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow (\lambda(b' : \mathbf{Bool}).\mathit{Bool})\ \mathrm{tt} =}}{\dfrac{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathbf{Bool} = (\lambda(b' : \mathrm{Bool}).\mathit{Boo}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathbf{Bool} = (\lambda(b' : \mathrm{Bool}).\mathit{Bool}}}$$

Example (Cont.)

- Similarly follows

$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathbf{Bool} = (\lambda(b' : \mathbf{Bool}).\mathit{Bool}$$

Example (Cont.)

– Using part of the proof above, we derive

$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow c : (\lambda(b' : \mathbf{Bool}).\mathit{Boo}$$

$$\dfrac{\dfrac{\dfrac{\cdots}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{Context}}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow c : \mathbf{Bool}} \quad \dfrac{\cdots}{b{:}\mathbf{Bool}, c{:}\mathbf{Bool} \Rightarrow \mathbf{Bool} = (\lambda(}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow c : (\lambda(b' : \mathbf{Bool}).\mathit{Bo}}$$

– We derive

$$b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{ff} : (\lambda(b' : \mathbf{Bool}).\mathit{Boo}$$

$$\dfrac{\dfrac{\dfrac{\cdots}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{Context}}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{ff} : \mathbf{Bool}} \quad \dfrac{\cdots}{b{:}\mathbf{Bool}, c{:}\mathbf{Bool} \Rightarrow \mathbf{Bool} = (\lambda(}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{ff} : (\lambda(b' : \mathbf{Bool}).\mathit{Bo}}$$

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Example (Cont.)

- We derive $b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow b : \mathbf{Bool}$ using part of th

$$\frac{\begin{array}{c}\cdots \\ b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{Context}\end{array}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow b : \mathbf{Bool}}$$

Example (Cont.)

– Finally we obtain our judgement (we stack the premises
of lack of space):

$$\frac{\dfrac{\begin{array}{r} b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \lambda(b' : \mathbf{Bool}).\mathit{Bool} : \mathbf{B} \\ b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow c : (\lambda(b' : \mathbf{Bool}).\mathit{B} \\ b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{ff} : (\lambda(b' : \mathbf{Bool}).\mathit{} \\ b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow b : \mathbf{Bool} \end{array}}{b : \mathbf{Bool}, c : \mathbf{Bool} \Rightarrow \mathrm{Case}_{\mathrm{Bool}}\ (\lambda(b' : \mathbf{Bool}).\mathit{Bo}}}{b : \mathbf{Bool} \Rightarrow \lambda(c : \mathbf{Bool}).\mathrm{Case}_{\mathrm{Bool}}\ (\lambda(b' : \mathbf{Bool}).\mathit{Bool})}$$

$$\overline{\lambda(b, c : \mathbf{Bool}).\mathrm{Case}_{\mathrm{Bool}}\ (\lambda(b' : \mathbf{Bool}).\mathit{Bool})\ c\ \mathrm{ff}\ b : \mathbf{Bo}}$$

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Elimination into Type

We can extend add elimination and equality rules, havin

Elimination Rule into Type

$$\frac{C : \mathrm{Bool} \rightarrow \mathrm{Type} \quad ic : C\ \mathrm{tt} \quad ec : C\ \mathrm{ff} \quad cond}{\mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec\ cond : C\ cond}$$

Equality Rules into Type

$$\frac{C : \mathrm{Bool} \rightarrow \mathrm{Type} \quad ic : C\ \mathrm{tt} \quad ec : C\ \mathrm{ff}}{\mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec\ \mathrm{tt} = ic : C\ \mathrm{tt}}$$

$$\frac{C : \mathrm{Bool} \rightarrow \mathrm{Type} \quad ic : C\ \mathrm{tt} \quad ec : C\ \mathrm{ff}}{\mathrm{Case}_{\mathrm{Bool}}\ C\ ic\ ec\ \mathrm{ff} = ec : C\ \mathrm{ff}}$$

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Elimination into Type (Cont.)

We can extend this into an elimination rule **into Kind or ot**

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Bool in Agda

- We introduce Bool by simply **listing its constructors** (s
syntax):

$$\mathrm{data\ Bool} = \mathrm{tt} \mid \mathrm{ff}$$

- This introduces as well constants
  - $\mathrm{tt} :: \mathrm{Bool}$
  - $\mathrm{ff} :: \mathrm{Bool}$

- With this syntax, each constructor **can occur at most on**
  - i.e. we cannot define a second type having constructor
  e.g. for defining True (which is used later):

$$\mathrm{data\ True} = \mathrm{tt}$$

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Bool in Agda (Cont.)

- The definition of Bool above is treated in Agda as an **ab** following three **more fundamental Agda definitions**:

$$
\begin{array}{rcl}
\mathrm{Bool} & :: & \mathrm{Set} \\
 & = & \mathrm{data\ tt} \mid \mathrm{ff} \\
\mathrm{tt} & : & \mathrm{Bool} \\
 & = & \mathrm{tt@Bool} \\
\mathrm{ff} & : & \mathrm{Bool} \\
 & = & \mathrm{ff@Bool}
\end{array}
$$

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Bool in Agda (Cont.)

- The definition of Bool as

$$\begin{array}{rcl} \mathrm{Bool} & :: & \mathrm{Set} \\ & = & \mathrm{data}\ \mathrm{tt} \mid \mathrm{ff} \end{array}$$

introduces Bool as a set **having constructors $\mathrm{tt@Bool}$**

– So tt, and ff **have to be defined separately**.
– If it is clear that the element in question is of type B
replace $\mathrm{tt@Bool}$ by $\mathrm{tt@_}$.
– The definition of Bool as above **doesn't prevent**
another set with constructors tt or ff.
– This syntax is the only one allowed, if one defines a s
keyword depending on arguments.
More about this later.

Bool in Agda (Cont.)

- **Internally**, tt will always be represented as $\mathrm{tt@Bool}$, simi
  - So Agda **evaluates tt to $\mathrm{tt@Bool}$**.
  - This can be seen when using for instance **"agda-c**
    compute weak head normal form.

Case Distinction

- Elimination in Agda is **based on case distinction**.
  - Assume we want to define
    - $f : \mathrm{Bool} \rightarrow \mathrm{Bool}$, s.t.
    - $f\ \mathrm{tt} = \mathrm{ff}$,
    - $f\ \mathrm{ff} = \mathrm{tt}$.
  - So we have the goal:

$$\begin{array}{lll} f & (x :: \mathrm{Bool}) & \\ & :: & \mathrm{Bool} \\ & = & \{!\ \ !\} \end{array}$$

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Case Distinction (Cont.)

- We can then type into the goal x and choose the menu it
  - This **introduces a case distinction** by the constructor u x:
    x could have been introduced as tt or ff.

- The goal expands to:

$$
\begin{array}{lll}
f & (x :: \mathrm{Bool}) & \\
 & :: & \mathrm{Bool} \\
 & = & \mathrm{case}\ x\ \mathrm{of} \\
 & & \quad \{\ (\mathrm{tt}) \rightarrow \{!\ \ !\}; \\
 & & \quad\ \ \ (\mathrm{ff}) \rightarrow \{!\ \ !\};\}
\end{array}
$$

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Case Distinction (Cont.)

- The **value of x** in the first goal **can be tested** as follows
  - Position the cursor in the first goal and choose (goal-) **"agda-compute-WHNF"**
    - **"Compute weak head normal form"** means essent **"compute the result of reducing that term"**.
      - More precisely this means that a term is reduced u constructor (or is a variable).
  - Then type into the mini-buffer x.
  - One gets the answer

    tt@_.

Case Distinction (Cont.)

- Alternatively, **check**, the cursor being in that goal, **the co**
  - (use goal-menu **"agda-context"**):

  It contains

  $x :: \mathrm{Bool} = \mathrm{tt}@_.$

- Similarly one finds that in the second goal x is $\mathrm{ff}@_$.

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Case Distinction (Cont.)

- Now we can solve the new goals by inserting
  - ff into the first one,
  - tt into the second one.
- We obtain a function:

$$\begin{array}{lll} f & (x :: \mathrm{Bool}) & \\ & :: & \mathrm{Bool} \\ & = & \mathrm{case}\ x\ \mathrm{of} \\ & & \qquad \{\ (\mathrm{tt}) \rightarrow \mathrm{ff}; \\ & & \qquad\ \ \ (\mathrm{ff}) \rightarrow \mathrm{tt}; \} \end{array}$$

- $f\ x$ is the **negation of x**.

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Testing the Defined Function

- We can test our function by using **"agda-compute-WHNF"**.
- We have to create a goal for this.
  - The reduction machinery is **context dependent**.
  - The context depends on where in the buffer we are.
    - See the above example where x was depending on th
  - Not every place in the buffer is a good place.
  - **Good places for context are goals**, and that's **the** Agda allows us to **compute the weak head normal fo**

Testing the Defined Function

- So we
  - type in a dummy goal:

$$\begin{array}{lcl} test & :: & \mathrm{Set} \\ & = & \{! \quad !\} \end{array}$$

  - move to the new goal
  - choose **"agda-compute-WHNF"**,
  - and type into the mini-buffer f tt.
- The result shown is $\mathrm{ff@_}$.

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(b) The Finite Sets

Bool can be generalized to sets having n elements (n a fixe

Formation Rule

$$\mathrm{Fin}_n : \mathrm{Set}$$

Introduction Rules

$$\mathrm{A}^n_k : \mathrm{Fin}_n$$

(for $k = 0, \ldots, n-1$)

Elimination Rule

$$\frac{\begin{array}{c} C : \mathrm{Fin}_n \longrightarrow \mathrm{Set} \\ s_0 : C\ \mathrm{A}^n_0 \\ s_1 : C\ \mathrm{A}^n_1 \\ \cdots \\ s_{n-1} : C\ \mathrm{A}^n_{n-1} \\ a : \mathrm{Fin}_n \end{array}}{\mathrm{Case}_n\ C\ s_0\ \ldots\ s_{n-1}\ a : C\ a}$$

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The Finite Sets (Cont)

Equality Rules

$$\frac{\begin{array}{c} C : \mathrm{Fin}_n \longrightarrow \mathrm{Set} \\ s_0 : C\ \mathrm{A}^n_0 \\ s_1 : C\ \mathrm{A}^n_1 \\ \cdots \\ s_{n-1} : C\ \mathrm{A}^n_{n-1} \end{array}}{\mathrm{Case}_n\ C\ s_0\ \ldots\ s_{n-1}\ \mathrm{A}^n_k = s_k : C\ \mathrm{A}^n_k}$$

(for $k = 0, \ldots, n-1$).

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Omitting Premises in Equality Rules

Since the premises of the equality rule can in most cases b
the introduction and elimination rules, we will **usually omi**
for instance for the previous rule:

$$\mathrm{Case}_n\ C\ s_0\ \ldots\ s_{n-1}\ \mathrm{A}^n_k = s_k : C\ \mathrm{A}^n_k$$

We sometimes even **omit the type**:

$$\mathrm{Case}_n\ C\ s_0\ \ldots\ s_{n-1}\ \mathrm{A}^n_k = s_k$$

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More compact elimination rules

- $$\begin{aligned}\mathrm{Case}_n : (C : \mathrm{Fin}_n \to \mathrm{Set}) \\ \to (s_0 : C\ \mathrm{A}^n_0) \\ \to \cdots \\ \to (s_{n-1} : C\ \mathrm{A}^n_{n-1}) \\ \to (a : \mathrm{Fin}_n) \\ \to C\ a\end{aligned}$$

.

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Elimination into Type

- Similarly as for Bool we can write down **elimination rule** $\mathbf{C : Fin_n \to Type}$ (instead of $C : \mathrm{Fin}_n \to \mathrm{Set}$).

- This can be done for all sets defined later as well.

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Rules for True

True is the special case Fin_n for $n = 1$:

Formation Rule

$$\mathrm{True} : \mathrm{Set}$$

Introduction Rules

$$\mathrm{true} : \mathrm{True}$$

Elimination Rule

$$\frac{C : \mathrm{True} \to \mathrm{Set} \quad c : C\ \mathrm{true} \quad t : \mathrm{True}}{\mathrm{Case}_{\mathrm{True}}\ c\ t : C\ t}$$

Equality Rule

$$\frac{C : \mathrm{True} \to \mathrm{Set} \quad c : C\ \mathrm{true}}{\mathrm{Case}_{\mathrm{True}}\ c\ \mathrm{true} = c : C\ \mathrm{true}}$$

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Rules for True (Cont.)

- $\mathrm{Case}_{\mathrm{True}}$ is **computationally not very interesting**.
  - $\mathrm{Case}_{\mathrm{True}}\ c$ is the untyped function $\lambda x.c$.
  - However, in Agda we might not be able to derive

$$\lambda(t : \mathrm{True}).c : (t : \mathrm{True}) \to C\ t$$

- From a **logic point of view**, it expresses:
  From an element of $C\ \mathrm{true}$ we obtain an element of $C\ t$ **f**
  - So there is no $C : \mathrm{True} \to \mathrm{Set}$ s.t. $C\ \mathrm{true}$ is inhabit
    inhabited for some other $x : \mathrm{True}$.
  - This means that all elements of x of type True are **indis**
    true, i.e. they are **identical to true**.
  - This equality is called Leibnitz equality).

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Rules for False

False is the special case Fin_n for $n = 0$:

Formation Rule

$$\mathrm{False} : \mathrm{Set}$$

There is no Introduction Rule

Elimination Rule

$$\frac{C : \mathrm{False} \to \mathrm{Set} \qquad f : \mathrm{False}}{\mathrm{Case}_{\mathrm{False}}\ f : C\ f}$$

There is no Equality Rule

n

False

- False has **no elements**.
- It is formula which is **always false**.
  - As well called **absurdity**.
- $\mathrm{Case}_{\mathrm{False}}$ expresses: **from an element f of False we o**
  of any set (which might depend on f).
  - From a logic point of view this is **"Ex falsum quo**
    absurdity follows anything).
    E.g. A **false formula** like "$0 = 1$" or "Swansea lies in
    everything.

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False (Cont.)

- $\mathrm{Case}_{\mathrm{False}}$ has **no computational meaning**, since there is
be applied to.
  - Applies of course only if we are working in a **terminati**
  - If we had **full recursion**, we could define $f : \mathrm{False}$ by f
    However that f doesn't reduce to canonical form.
  - That's why it's important to carry out the **terminatio**
    otherwise one obtains for instance elements of False

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Finite Sets in Agda

- **Finite sets** can be introduced by giving **one constructor**
  E.g.

$$\text{data Colour} = \text{blue} \mid \text{red} \mid \text{green}$$

  - With this we obtain $\text{red} :: \text{Colour}$
  - And we can define for instance

$$\begin{array}{lll} \text{is_red} & (c :: \text{Colour}) & \\ & :: & \text{Bool} \\ & = & \text{case } c \text{ of} \\ & & \quad \{ \; (\text{red}) \;\; \rightarrow \;\; \text{tt}; \\ & & \quad\;\;\; (\text{green}) \;\; \rightarrow \;\; \text{ff}; \\ & & \quad\;\;\; (\text{blue}) \;\; \rightarrow \;\; \text{ff}; \} \end{array}$$

False in Agda

- In Agda we can define the empty set as a "data"-set **with**

$$\text{data False} =$$

- If we want to solve

$$\begin{array}{lll} g & (x :: \text{False}) \\ & :: & \text{Bool} \\ & = & \{! \quad !\} \end{array}$$

we can insert into the goal x and choose menu-item **"ag**

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False in Agda (Cont.)

- The result is

$$\begin{array}{lll} g & (x :: \mathrm{False}) & \\ & :: & \mathrm{Bool} \\ & = & \mathrm{case}\ x\ \mathrm{of}\ \{\ \ \} \end{array}$$

- If we make case distinction on x there is **no case to ch**
don't have to define anything.

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Example for the Use of False

- Assume the **type of trees**:

$$\text{data Tree} = \text{pine} \mid \text{oak}$$

- Below we will show, how to introduce a function

$$\text{IsOak} :: \text{Tree} \to \text{Set}$$

s.t.

$$\begin{array}{rcl} \text{IsOak pine} & = & \text{False} \\ \text{IsOak oak} & = & \text{True} \end{array}$$

Example for the Use of False (Cont.)

- If we want to define a function from trees, which are oak set, we can do so by requiring **an additional argument** '

$$
\begin{array}{lll}
f & (t :: \mathrm{Tree}) & \\
  & (p :: \mathrm{IsOak\ t}) & \\
  & :: \quad A & \\
  & = \quad \mathrm{case}\ t\ \mathrm{of} & \\
  & \qquad \{\ \mathrm{pine} \ \rightarrow\ \mathrm{case}\ p\ \ \mathrm{of}\ \{\ \ \} & \\
  & \qquad \quad \mathrm{oak} \ \rightarrow\ \cdots ; \} &
\end{array}
$$

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Example for the Use of False (Cont.)

- In order to use f we have to **know** that t is an oak tree,
  - i.e. we have to provide an argument p which expresse
    know this.
- Note that we **don't have to invent a result** of f in case

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Example 2 for the Use of False

- Similarly we can introduce a **stack**, together with a predic

$$\text{NotEmpty} :: \text{Stack} \rightarrow \text{Set}$$

s.t.

$$\text{NotEmpty}\ s = \text{False}$$

if s is the empty stack.

- Now we can define

$$\begin{array}{ll} \text{pop} & (s :: \text{Stack}) \\ & (p :: \text{NotEmpty}\ s) \\ & ::\quad \text{Stack} \\ & =\quad \cdots \end{array}$$

- Again we **don't have to provide a result, in case s is e**

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True in Agda

- The definition of True in Agda is **straightforward**:

$$\text{data False} = \text{true}$$

- Case distinction will require to **solve the case true**:

$$\begin{array}{lll} g & (x :: \text{True}) & \\ & :: & \text{Bool} \\ & = & \text{case } x \text{ of } \{(\text{true}) \rightarrow \{! \quad !\};\} \end{array}$$

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(c) Atomic Formulae and the Traffic Light Example

Atomic Formulae

- We have already introduced **two formulae**:
  - True.
    - True is **inhabited**.
      - There is a proof of it (true).
      - True is therefore **type-theoretically true**:
        A formula is **type-theoretically true**, if it is proval

Truth in type theory means provability

Atomic Formulae (Cont.)

- False.
  * False is **not inhabited**.
    - There is **no proof** of False.
      Furthermore, from any proof of False we can
      (elimination rules for False).
    - False is therefore **type-theoretically false**:
      A formula is **type-theoretically false**, if from
      everything.
    - Since this implies that we can derive False and f
      derive everything, this is equivalent to the following
    - A formula is **type-theoretically false**, if from a
      derive False (i.e. a **contradiction**).

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Atomic Formulae (Cont.)

- There are formulae in type theory, which are **neither type-** **nor type-theoretically false**.
  - This means that we can neither prove them, nor deri contradiction.
  - Truth in type theory means that we **know that it is tr**
  - Falsity in type theory means that we **know that it can**
  - There are formulae in type theory for which **neither of**
- True and False as above are formulae corresponding to **true and false**.

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atom

- We can **map** truth values to their corresponding formula:

$$
\begin{aligned}
\mathrm{atom} &: \mathrm{Bool} \rightarrow \mathrm{Set} \\
\mathrm{atom}\ \mathrm{tt} &= \mathrm{True} \\
\mathrm{atom}\ \mathrm{ff} &= \mathrm{False}
\end{aligned}
$$

- This can be defined using **case distinction**.

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atom (Cont.)

- This corresponds to the following rules (which are not nee

$$\frac{b : \mathrm{Bool}}{\mathrm{atom}\ b : \mathrm{Set}}$$

$$\mathrm{atom\ tt} = \mathrm{True}$$
$$\mathrm{atom\ ff} = \mathrm{False}$$

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atom in Agda

$$
\begin{array}{llll}
\text{atom} & (b :: \text{Bool}) & & \\
 & :: \quad \text{Set} & & \\
 & = \quad \text{case } b \text{ of} & & \\
 & \qquad\qquad \{ \ (\text{tt}) & \rightarrow & \text{True;} \\
 & \qquad\qquad \phantom{\{} \ (\text{ff}) & \rightarrow & \text{False;} \}
\end{array}
$$

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Decidable Predicates

- Using atom we can now define **decidable predicates** on

- Assume we have a **set of states** of a system A.
  - E.g. the set of states a railway controller can choose.

- Assume we have a function $f : A \to \mathrm{Bool}$.
  - E.g. $f\ a$ means: **state a is safe**.

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Decidable Predicates (Cont.)

- Let now $g : A \rightarrow \mathrm{Set}$, $g\ a = \mathrm{atom}(f\ a)$.
  - If $f\ a$ is **true** (e.g. a is safe), $g\ a$ is **inhabited**.
  - If $f\ a$ is **false** (e.g. a is unsafe), $g\ a$ is **not inhabited**.
- Now, the existence of a $h : (a : A) \rightarrow g\ a$ means:
  - For all $a : A$ we have $g\ a$ is **inhabited**,
  - ie. for all $a : A$, $f\ a$ is **true**,
  - e.g. for all $a : A$, a is **safe**.

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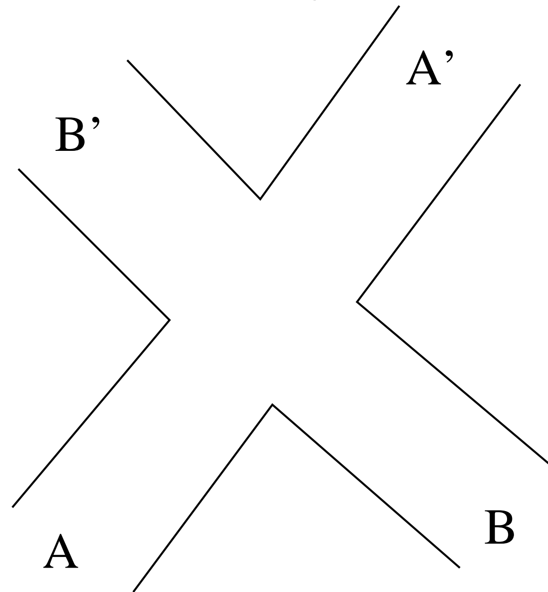
The Traffic Light Example

- Assume a **road crossing**, controlled by **traffic lights**:



- Assume from each direction A, A', B, B' there is one traf
  - but A and A' always coincide, similarly B and B'.

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The Set of Physical States

- For simplicity assume that **each traffic light is either re**

$$\text{data Colour} = \text{red} \mid \text{green}$$

- The set of **physical states of the system** is given by a pa
  colour of A (and therefore as well A') and of B (and B')

$$\begin{array}{lll} \text{Phys_State} & :: & \text{Set} \\ & = & \text{sig}\{ \begin{array}{lll} \text{sigA} & :: & \text{Colour;} \\ \text{sigB} & :: & \text{Colour;} \end{array} \end{array}$$

The Set of Control States

- The set of **control states** is a set of states of the system, system can choose.
  - Each of these states **should be safe**.
  - In our example, **all safe states will be captured** (this can usually be only achieved in small examples).
- A **complete set of control states** consists of:
  - Allred – all signals are red.
  - Agreen – signal A (and A') is green, signal B is red.
  - Bgreen – signal B is green, signal A is red.

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The Set of Control States (Cont.)

- We therefore define

$$\text{data Control_State} = \text{Allred} \mid \text{Agreen} \mid \text{Bg}$$

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Mapping Control States to Physical Sta

- We define the **state of signals A, B depending on a co**

$$
\begin{array}{lll}
\mathrm{toSigA} & (s :: \mathrm{Control_State}) & \\
 & :: & \mathrm{Colour} \\
 & = & \mathrm{case}\ s\ \mathrm{of} \\
 & & \quad \{\ (\mathrm{Allred}) \ \longrightarrow\ \mathrm{red}; \\
 & & \quad\ \ \ (\mathrm{Agreen}) \ \longrightarrow\ \mathrm{green} \\
 & & \quad\ \ \ (\mathrm{Bgreen}) \ \longrightarrow\ \mathrm{red}; \}
\end{array}
$$

$$
\begin{array}{lll}
\mathrm{toSigB} & (s :: \mathrm{Control_State}) & \\
 & :: & \mathrm{Colour} \\
 & = & \mathrm{case}\ s\ \mathrm{of} \\
 & & \quad \{\ (\mathrm{Allred}) \ \longrightarrow\ \mathrm{red}; \\
 & & \quad\ \ \ (\mathrm{Agreen}) \ \longrightarrow\ \mathrm{red}; \\
 & & \quad\ \ \ (\mathrm{Bgreen}) \ \longrightarrow\ \mathrm{green}
\end{array}
$$

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Mapping Control States to Physical Sta

- Now we can define the **physical state corresponding to**

$$
\begin{array}{lll}
\mathrm{phys_state} & (s :: \mathrm{Control_State}) & \\
 & :: & \mathrm{Phys_State} \\
 & = & \mathrm{struct}\{\ \mathrm{sigA} \ = \ \mathrm{toSigA} \\
 & & \qquad\qquad \mathrm{sigB} \ = \ \mathrm{toSigB}
\end{array}
$$

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Safety Predicate

- We define now **when a physical state is safe**:
  - It is **safe iff not both signals are green**.
  - We define now a corresponding predicate **directly**, with
    Boolean function.
  - We first define a predicate depending on two signals:

$$
\begin{array}{lll}
\mathrm{CorAux} & (a :: \mathrm{Colour}) & \\
 & (b :: \mathrm{Colour}) & \\
 & :: \quad \mathrm{Set} & \\
 & = \quad \mathrm{case}\ a\ \mathrm{of} & \\
 & \qquad \{\ (\mathrm{red}) \quad \longrightarrow \quad \mathrm{True}; & \\
 & \qquad\ \ \ (\mathrm{green}) \quad \longrightarrow \quad \mathrm{case}\ b\ \mathrm{of} & \\
 & \qquad\qquad\qquad \{\ (\mathrm{red}) & \\
 & \qquad\qquad\qquad\ \ \ (\mathrm{green}) &
\end{array}
$$

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Safety Predicate (Cont.)

– Now we define

$$\begin{array}{lll} \mathrm{Cor} & (s :: \mathrm{Phys_State}) & \\ & :: & \mathrm{Set} \\ & = & \mathrm{CorAux}\ s.\mathrm{sigA}\ s.\mathrm{sigB} \end{array}$$

- **Remark:** In some cases in order to define a function from s
a sig-set) into some other set, it is better first to **intro
function**, depending on the components of that product.
  – In the current example this wouldn't have caused prob
  complex examples it does (due to the lack of the η-rule

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Safety of the System

- Now we show that **all control states are safe**:

$$
\begin{array}{lll}
\text{cor_proof} & (s :: \text{Control_State}) & \\
 & :: \quad \text{Cor}(\text{phys_state}\ s) & \\
 & = \quad \text{case}\ s\ \text{of} & \\
 & \qquad \{\ (\text{Allred}) & \rightarrow\ \text{true} \\
 & \qquad\ \ (\text{Agreen}) & \rightarrow\ \text{true} \\
 & \qquad\ \ (\text{Bgreen}) & \rightarrow\ \text{true}
\end{array}
$$

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Safety of the System (Cont.)

- The first element true was an element of **Cor(phys_sta** reduces to **True**.
- Similarly for the other two elements.
- This works only because **each control state correspo** **physical state**.
  - If this hadn't been the case, we would have gotten ir goal to solve is **False**, which we can't solve.

Safety of the System (Cont.)

- If one makes a **mistake** which results in an unsafe situatio
  (e.g. sets toSigB Agreen = green, then in the last step w
  of type False.
  - Then we can't solve this goal directly and **cannot prove**
  - (In fact we could type-theoretically solve this goal by us
    (e.g. solve this goal as **cor_proof Agreen**), but this
    by the termination check.)

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(d) The Disjoint Union of Sets

Formation Rule

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set}}{A + B : \mathrm{Set}}$$

Introduction Rules

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad a : A}{\mathrm{inl}\ A\ B\ a : A + B}$$

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad b : B}{\mathrm{inr}\ A\ B\ b : A + B}$$

Elimination Rule

$$\frac{\begin{array}{c} A : \mathrm{Set} \\ B : \mathrm{Set} \\ C : (A + B) \to \mathrm{Set} \\ sl : (a : A) \to C\ (\mathrm{inl}\ A\ B\ a) \\ sr : (b : B) \to C\ (\mathrm{inr}\ A\ B\ b) \\ d : A + B \end{array}}{\mathrm{Plus_Split}\ A\ B\ C\ sl\ sr\ d : C\ d}$$

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The Disjoint Union of Sets (Cont.)

Equality Rules

$$
\begin{array}{lcl}
\text{Plus_Split}\ A\ B\ C\ sl\ sr\ (\text{inl}\ A\ B\ a) & = & sl\ a : C\ (\text{i} \\
\text{Plus_Split}\ A\ B\ C\ sl\ sr\ (\text{inr}\ A\ B\ b) & = & sr\ b : C\ (\text{i}
\end{array}
$$

Disjoint Union using the Logical Framework

- A **more compact notation** is:
  - $(+) : \mathrm{Set} \to \mathrm{Set} \to \mathrm{Set}$, written infix.
  - $\mathrm{inl} : (A, B : \mathrm{Set}) \to A \to (A + B)$.
  - $\mathrm{inr} : (A, B : \mathrm{Set}) \to B \to (A + B)$.
  - $\mathrm{Plus_Split} : (A, B : \mathrm{Set})$

$$
\begin{aligned}
&\to (C : (A + B) \to \mathrm{Set})\\
&\quad \to (sl : (a : A) \to C\ (\mathrm{inl}\ A\ B\ a))\\
&\quad\quad \to (sr : (b : B) \to C\ (\mathrm{inr}\ A\ B\ b))\\
&\quad\quad\quad \to (d : A + B)\\
&\quad\quad\quad\quad \to C\ d\ \ .
\end{aligned}
$$

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Disjoint Union in Agda

- The disjoint union can be defined as a "data"-set having inl (in-left) and inr (inright):

$$
\begin{array}{ll}
(+) & (A :: \mathrm{Set}) \\
& (B :: \mathrm{Set}) \\
& :: \quad \mathrm{Set} \\
& = \quad \mathrm{data}\ \mathrm{inl}(a :: A) \mid \mathrm{inr}(b :: B)
\end{array}
$$

Disjoint Union in Agda (Cont.)

- The notation $(+)$ means, that $+$ can be used **infix**.
- Now we have, if $A, B :: \mathrm{Set}$:
  - $\mathrm{inl}@(A + B) :: A \to (A + B)$
  - $\mathrm{inr}@(A + B) :: B \to (A + B)$
  - This can be checked using the menu "**agda-infer-type**"
  - Note that we cannot assign a type to $\mathrm{inr}@_$.
  - $(+)$ **cannot** be defined using the abbreviated data nota
    (which would be of the form
    $\mathrm{data}\,(+) = \cdots$).

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Disjoint Union in Agda (Cont.)

- Elimination is again represented by case distinction.
  So if want to define for $A, B :: \mathrm{Set}$ for instance

$$\begin{array}{lll} f & (c :: A + B) \\ & :: & \mathrm{Bool} \\ & = & \{! \quad !\} \end{array}$$

  we can type into the goal c and choose menu "agda-case"

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Disjoint Union in Agda (Cont.)

- We obtain

$$
\begin{array}{lll}
f & (c :: A + B) & \\
 & :: & \text{Bool} \\
 & = & \text{case } c \text{ of} \\
 & & \quad \{ \ (\text{inl } a) \ \rightarrow \ \{! \ \ !\}; \\
 & & \quad \ \ \ (\text{inr } b) \ \rightarrow \ \{! \ \ !\}; \}
\end{array}
$$

and insert into the first goal e.g. true and the second one

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Use of Concrete Disjoint Sets

- It is usually **more convenient** to define concrete disjoint u more intuitive names for constructors, e.g.

$$\text{data Plant} = \text{tree}(t :: \text{Tree}) \mid \text{flower}(f :: \text{Fl}$$

- Now one can define for instance

$$\begin{array}{lll} \text{isFlower} & (p :: \text{Plant}) & \\ & :: & \text{Bool} \\ & = & \text{case } p \text{ of} \\ & & \quad \{ \ (\text{tree } t) \quad \rightarrow \quad \text{ff}; \\ & & \quad \ \ \ (\text{flower } f) \quad \rightarrow \quad \text{tt}; \end{array}$$

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(e) The Σ-Set

Formation Rule

$$\frac{A : \mathrm{Set} \qquad B : A \to \mathrm{Set}}{\Sigma\ A\ B : \mathrm{Set}}$$

Introduction Rule

$$\frac{\begin{array}{c} A : \mathrm{Set} \\ B : A \to \mathrm{Set} \\ a : A \\ b : B\ a \end{array}}{\mathrm{p}\ A\ B\ a\ b : \Sigma\ A\ B}$$

Elimination Rule

$$\frac{\begin{array}{c} A : \mathrm{Set} \\ B : A \to \mathrm{Set} \\ C : (\Sigma\ A\ B) \to \mathrm{Set} \\ s : (a : A) \to (b : B\ a) \to C\ (\mathrm{p}\ A\ B\ a\ b)) \\ d : \Sigma\ A\ B \end{array}}{\mathrm{Sigma_Split}\ A\ B\ C\ s\ d : C\ d}$$

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The Σ-Set (Cont)

Equality Rule

$$\mathrm{Sigma_Split}\ A\ B\ C\ s\ (\mathrm{p}\ A\ B\ a\ b) = s\ a\ b : C\ (\mathrm{p}\ \ldots$$

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The Σ-Set using the Logical Framewor

- The more compact notation is:
  - $\Sigma : (A : \mathrm{Set}) \to (B : A \to \mathrm{Set}) \to \mathrm{Set}$.
  - $\mathrm{p} : (A : \mathrm{Set}) \to (B : A \to \mathrm{Set}) \to (a : A) \to (b : B\ a) \to \Sigma\ A\ B$.

The Σ-Set using the Logical Framework (C

- Sigma_Split

$$
\begin{aligned}
&: (A : \mathrm{Set}) \\
&\quad \rightarrow (B : A \rightarrow \mathrm{Set}) \\
&\qquad \rightarrow (C : (\Sigma\ A\ B) \rightarrow \mathrm{Set}) \\
&\qquad\quad \rightarrow (s : (a : A, b : B\ a) \\
&\qquad\qquad\qquad \rightarrow C\ (\mathrm{p}\ A\ B\ a\ b)) \\
&\qquad\qquad \rightarrow (d : \Sigma\ A\ B) \\
&\qquad\qquad\quad \rightarrow C\ d\ \ .
\end{aligned}
$$

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The Σ-Set and the Dependent Produc

- The **dependent product and the Σ-set are very simila**
  - Both have similar introduction rules (for the Σ-set, the additional arguments A, B necessary for bureaucratic r
  - One can define the projections π_0, π_1 using $\mathrm{Sigma_Spl}$

$$\begin{array}{lcl}\pi_0 & = & \mathrm{Sigma_Split}\ A\ B\ (\lambda x.A)\ (\lambda(x : A).\lambda(y :\\ \pi_1 & = & \mathrm{Sigma_Split}\ A\ B\ (\lambda x.B\ \pi_0(x))\ (\lambda(x : A)\end{array}$$

  - On the other hand, from π_0, π_1 we can define $\mathrm{Sigma_S}$

$$\lambda A, B, C, s, d.s\ \pi_0(d)\ \pi_1(d)\ .$$

The Σ-Set and the Dependent Produc

- However the dependent product has the η-**rule** (whic implemented in Agda).
- Because of the lack of η-rule, Σ works usually **better tha product** in Agda.
  - I personally **don't use the dependent product** of Agd

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The Σ-Set in Agda

- Σ can be defined as a "data"-set with constructor p:

$$\begin{array}{lll} \text{Sigma} & (A :: \text{Set}) & \\ & (B :: A \rightarrow \text{Set}) & \\ & :: & \text{Set} \\ & = & \text{data}\ p\ (a :: A)\ (b :: B a) \end{array}$$

The Σ-Set in Agda (Cont.)

- Again one usually defines concrete Σ-sets more directly.
- **Example:** Assume we have defined
  - a set $\mathrm{Plant_Group}$ for **groups of plants** (e.g. "tree",
  - depending on $g :: \mathrm{Plant_Group}$, sets $\mathrm{Plants_in_group}$
    plants in that group.
- The **set of plants** can then be defined as

$$\mathrm{data\ Plant} = \mathrm{plant}\ (g :: \mathrm{Plant_Group})(pg :: \mathrm{Plants}$$

The Σ-Set in Agda (Cont.)

- Not surprisingly, for **elimination** we use **case distinction**

$$
\begin{array}{lll}
f & (p :: \mathrm{Plant}) & \\
 & :: & \mathrm{Plant_group} \\
 & = & \mathrm{case}\ p\ \mathrm{of} \\
 & & \quad \{\ (\mathrm{plant}\ g\ pg) \ \rightarrow\ g; \}
\end{array}
$$

Formulae in Dependent Type Theo

- We have seen how to represent **atomic decidable formu**
- Now treatment of complex formulae constructed using **log**

Conjunction

- $\mathbf{A} \wedge \mathbf{B}$ is true iff both A is true and B is true.
- Therefore a proof of $A \wedge B$ consists of a **proof of A and**
  - It is therefore a pair $\langle p, q\rangle$ consisting of a proof p of A a
- Therefore the set of proofs of $A \wedge B$ is the set of pairs of B, i.e. $\mathbf{A} \times \mathbf{B}$.
- We can **identify** $A \wedge B$ with $A \times B$.

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Conjunction (Cont.)

- With this identification, the **introduction rule** for $\wedge$ allo of $A \wedge B$ from a proof of A and a proof of B:

$$\frac{p : A \qquad q : B}{\langle p, q \rangle : A \wedge B}$$

- This means that we can **derive A $\wedge$ B from A and B**.

- This is what is expressed by the **ordinary introduction ru**

$$\frac{A \qquad B}{A \wedge B}$$

Conjunction (Cont.)

- The **elimination rule** for $\wedge$ allows to project a proof of A
  A and a proof of B:

$$\frac{p : A \wedge B}{\pi_0(p) : A} \qquad \frac{p : A \wedge B}{\pi_1(p) : B}$$

- This means that we can **derive from A $\wedge$ B both A anc**

- This is what is expressed by the **ordinary elimination rul**

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

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Disjunction

- $A \lor B$ is true iff A is true or B is true.

- Therefore a **proof of A $\lor$ B consists of a proof of A** **plus the information which one.**
  - It is therefore an element $\mathrm{inl}\, p$ for a proof $p : A$ or an proof $q : B$.

- Therefore the set of proofs of $A \lor B$ is the **disjoint unio** $\mathbf{A + B}$.

- We can **identify** $A \lor B$ with $A + B$.

Disjunction (Cont.)

- With this identification, the **introduction rules** for $\vee$ allo of $A \vee B$ from a proof of A or from a proof of B.

$$\frac{A : \mathrm{Set} \qquad B : \mathrm{Set} \qquad p : A}{\mathrm{inl}\ A\ B\ p : A + B} \qquad \frac{A : \mathrm{Set} \qquad B}{\mathrm{inr}\ A\ B\ p}$$

  – Omitting the premises $A, B : \mathrm{Set}$ and omitting them a and inr (which is needed only for bureaucratic reasons)

$$\frac{p : A}{\mathrm{inl}\ p : A + B} \qquad \frac{p : B}{\mathrm{inr}\ p : A + B}$$

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Disjunction (Cont.)

- This means that we can **derive A ∨ B from A and from**
- This is what is expressed by the **ordinary introduction ru**

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

Disjunction (Cont.)

- The **elimination rule** for $+$ allows to form from an ele[...] element of any set C provided we can compute such an el[...] from B:

$$\frac{\begin{array}{c} A : \mathrm{Set} \\ B : \mathrm{Set} \\ C : (A \vee B) \rightarrow \mathrm{Set} \\ sl : (a : A) \rightarrow C\ (\mathrm{inl}\ A\ B\ a) \\ sr : (b : B) \rightarrow C\ (\mathrm{inr}\ A\ B\ b) \\ d : A \vee B \end{array}}{\mathrm{Plus_Split}\ A\ B\ C\ sl\ sr\ d : C\ d}$$

- Omitting the dependency of C on $A \vee B$ and omittin[...] premises and arguments A, B and C we get:

$$\frac{d : A \vee B \qquad sl : A \rightarrow C \qquad sr : B \rightarrow [...]}{\mathrm{Plus_Split}\ sl\ sr\ d : C}$$

Disjunction (Cont.)

- This means that we can **derive from A $\vee$ B a formula C**
  C from A and from B.

- This is what is expressed by the **ordinary elimination rul**

$$\frac{A \vee B \quad \begin{matrix} A \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B \\ \vdots \\ C \end{matrix}}{C}$$

- (Note that in the ordinary elimination rule, from the pre
  from A" we obtain "$A \to C$", similarly for "C derivabl
  $B \to C$.)

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Implication

- We write temporarily $\supset$ for logical implication, in order to
  the function type $\rightarrow$.
  - Below we see that $\supset$ can be identified with $\rightarrow$.
- $A \supset B$ is true iff, whenever A is true then B is true.
- Therefore **if there is a proof of A, there must be a pr**
- Therefore a proof of $A \supset B$ is a **function, which takes**
  computes a proof of B.
- Therefore the set of proofs of $A \supset B$ is the **function typ**
- We can **identify** $A \supset B$ with $A \rightarrow B$.

Implication (Cont.)

- With this identification, the **introduction rule for $\supset$** allo of $A \supset B$ from a proof of B depending on a proof p of A

$$\frac{p : A \Rightarrow q : B}{\lambda(p : A).q : A \supset B}$$

- This means that, if we, **from assumptions p:A can prov**
  - (i.e. we can make use of a context $p : A$ for proving q :

  then we can derive A $\supset$ B without assuming p:A.

Implication (Cont.)

- This is what is expressed by the **ordinary introduction ru**

$$\frac{\begin{array}{c}A\\ \vdots\\ B\end{array}}{A \supset B}$$

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Implication (Cont.)

- The **elimination rule for $\supset$** allows to apply a proof p of
of q of A in order to obtain a proof of B:

$$\frac{p : A \supset B \qquad q : A}{p\,q : B}$$

- This means that we can **derive from A $\supset$ B and A that**

- This is what is expressed by the **ordinary elimination rul**

$$\frac{A \supset B \qquad A}{B}$$

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Negation

- $\neg \mathbf{A}$ has the same meaning as $\mathbf{A} \supset \bot$ (where $\bot$ is **absurdity** or the set False):
  - If there is no proof of A, then we can prove $A \supset \bot$.
  - If from any proof of A we can create a proof of abs
    cannot be a proof of A, A must be false.
- Therefore we can identify $\neg \mathbf{A}$ with $\mathbf{A} \to \mathbf{False}$.

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Universal Quantification

- Since we have many types, we have to write when using q
the type, the bound variable is ranging over:
We write therefore $\mathbf{\forall x : A.B}$, $\mathbf{\exists x : A.B}$.

- $\forall x : A.B$ is true iff, for all $x : A$ there exists a proof of B

- Therefore a proof of $\forall x : A.B$ is a **function, which t**
computes an element of B.

- Therefore the set of proofs of $\forall x : A.B$ is the **depende**
$\mathbf{(x : A) \to B}$.

- We can **identify** $\mathbf{\forall x : A.B}$ with $\mathbf{(x : A) \to B}$.

Universal Quantification (Cont.)

- With this identification, the **introduction rule** for $\forall$ allo
of $\forall x : A.B$ from a proof of B depending on an element

$$\frac{x : A \Rightarrow p : B}{\lambda(x : A).p : \forall x : A.B}$$

- This means that, if we, **from x:A can prove B**, then
$\forall x : A.B$ which doesn't depend on $x : A$.

Universal Quantification (Cont.)

- This is what is expressed by the **ordinary introduction ru**

$$\frac{B}{\forall x : A.B}$$

where

– **x might not occur free in any assumption of the p**
  * This is guaranteed in type theory, since $x : A$ must b
    of the context, so any other assumptions must be loc
    can therefore **not depend on x:A**.

– The **conclusion will no longer depend on free varial**
  * This corresponds in type theory to the fact that **x:/**
    occur in the context of the conclusion.

Universal Quantification (Cont.)

- The **elimination rule** for the dependent function type allo
  p of $\forall x : A.B$ to an element $a : A$ in order to obtain a pro

$$\frac{p : \forall x : A.B \qquad a : A}{p\, a : B[x := a]}$$

- This means that we can **derive from ∀x:A.B and an ele**
  B[x:=a] holds.

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Universal Quantification (Cont.)

- This is what is expressed by the **ordinary elimination rul**
  - For the simple languages used in ordinary logic, there i[s] that $a : A$; in more **complex type theories we have** **derivation**.

$$\frac{\forall x : A.B \qquad a : A}{B[x := a]}$$

Existential Quantification

- $\exists x : A.B$ is true iff there exists an $a : A$ such that $B[x :=$
- Therefore a proof of $\exists x : A.B$ is a **pair** $\langle \mathbf{a}, \mathbf{p} \rangle$ **consisti** $\mathbf{a} : \mathbf{A}$ **and a proof p of** $\mathbf{B}[\mathbf{x} := \mathbf{a}]$.
- Therefore the set of proofs of $\exists x : A.B$ is the **dependent** $(\mathbf{x} : \mathbf{A}) \times \mathbf{B}$.
- We can **identify** $\exists \mathbf{x} : \mathbf{A}.\mathbf{B}$ with $(\mathbf{x} : \mathbf{A}) \times \mathbf{B}$.

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Existential Quantification (Cont.)

- With this identification, the **introduction rule** for $\exists$ allo
of $\exists x : A.B$ from an element $a : A$ and a proof $p : B[x :=$

$$\frac{a : A \qquad p : B[x := a]}{\langle a, p \rangle : \exists x : A.B}$$

- This is what is expressed by the **ordinary introduction ru**

$$\frac{a : A \qquad B[x := a]}{\exists x : A.B}$$

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Existential Quantification (Cont.)

- The **elimination rule** for the dependent product allows t of $\exists x : A.B$ to an element $\pi_0(p) : A$ and proof $\pi_1(p) : B[$

- This kind of rule works only if we have **explicit proofs**.

- From this we can derive a rule which is essentially th deduction (in which one doesn't have explicit proofs):
  - Assume:
    * $C : \mathrm{Set}$, which does not depend on $x : A$,
    * $p : \exists x : A.B$ and
    * $x : A, y : B \Rightarrow c : C$.
  - Then we have $\mathbf{c[x := \pi_0(p), y := \pi_1(p)] : C}$, **not depending on x:A or y:B**.

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Existential Quantification (Cont.)

- Therefore the **rule in natural deduction** follows from rules:

$$\dfrac{\exists x : A.B \qquad \begin{matrix} x : A \\ B \\ \vdots \\ C \end{matrix}}{C}$$

where the conclusion does not depend on $x : A$ and B.

Constructive (or Intuitionistic) Logic

- From type theoretic proofs we can **directly extract progr**
- For instance, if $p : \forall x : A.\exists y : B.C(x, y)$, then we have
  - for $x : A$ it follows $b := \pi_0(p\ x) : B$ and $\pi_1(p\ x) : C(x,$
  - Therefore $f := \lambda x : A.\pi_0(p\ x)$ is a **function** $\mathbf{A \to B}$, a

$$\lambda(\mathbf{x} : \mathbf{A}).\pi_1(\mathbf{p}\ \mathbf{x}) : \forall \mathbf{x} : \mathbf{A}.\mathbf{C}(\mathbf{x}, \mathbf{f}\ \mathbf{x})$$

  i.e. we have a proof that $\forall \mathbf{x} : \mathbf{A}.\mathbf{C}(\mathbf{x}, \mathbf{f}\ \mathbf{x})$ **holds**.
  - Therefore, from a proof of $\forall x : A.\exists y : B.C(x, y)$, **function**, which computes the y from the x.

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Constructive Logic (Cont.)

- We can derive as well a function which **depending on p**
whether p= inl(a) or p = inr(b).

- Therefore we can decide, from a proof of a disjunctio
disjuncts holds.

- Now:
  - Any function in type theory is **recursive**.
  - We **cannot decide the Turing Halting problem**, i.e.
for a Turing machine whether it halts or not.
  - Therefore **we cannot prove in type theory**

$$\forall \mathbf{x} : \mathbf{Turing_Machine}.(\mathbf{x\ halts} \vee \neg(\mathbf{x\ h}$$

Constructive Logic (Cont.)

- In classical logic we **can prove the above**, since we ca
(tertium non datur) for any formula A.

- In type theory, this law **cannot hold**, unless we don't wan
can be evaluated.

  – The logic of type theory is **intuitionistic (constructiv**
  $A \vee \neg A$ and $\neg\neg A \rightarrow A$ don't hold for all formulae A.

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Constructive Logic (Cont.)

- In **classical logic**,
  - **$\exists x : A.B$ is equivalent to $\neg\forall x : A.\neg B$**,
  - **$A \vee B$ is equivalent to $\neg(\neg A \wedge \neg B)$**.

- If we take decidable atomic formulae only and replace $\exists x$ by the above formulae, then **all formulas provable in c derivable**.
  - This requires $(\neg\neg A) \to A$, which can be shown for all f decidable atomic formulae using $\neg$, $\to$, $\wedge$, $\forall$.
  - The formula $A \vee \neg A$ translates into $\neg(\neg A \wedge \neg\neg A)$, w since $\neg A$ and $\neg\neg A$ implies $\bot$.

- In this sense, **type theory contains classical logic**, but has as well so called **strong disjunction and existential**

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Constructive Logic (Cont.)

- Proof (using classical logic) of

$$\exists \mathbf{x} : \mathbf{A}.\mathbf{B} \leftrightarrow \neg \forall \mathbf{x} : \mathbf{A}.\neg \mathbf{B} \quad :$$

  - We have classically:

$$\neg\neg \mathbf{A} \rightarrow \mathbf{A} \quad :$$

    * If A is true, then $\neg\neg A \rightarrow A$ holds.
    * If A is false, then $\neg\neg A$ is false, therefore $\neg\neg A \rightarrow A$

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Constructive Logic (Cont.)

- We show intuitionistically $\neg(\exists \mathbf{x} : \mathbf{A}.\mathbf{B}) \leftrightarrow \forall \mathbf{x} : \mathbf{A}.\neg\mathbf{B}$:
  - Assume $\neg(\exists x : A.B)$, $x : A$ and show $\neg B$.
    If we had B, then we had $\exists x : A.B$, contradicting $\neg(\exists x$
    $\neg B$.
  - Assume $\forall x : A.\neg B$. Show $\neg(\exists x : A.B)$:
    Assume $\exists x : A.B$. Assume x s.t. B holds.
    By $\forall x : A.\neg B$ we get $\neg B$, therefore a contradiction.

- Now it follows (classically):

$$(\exists \mathbf{x} : \mathbf{A}.\mathbf{B}) \leftrightarrow \neg\neg(\exists \mathbf{x} : \mathbf{A}.\mathbf{B}) \leftrightarrow \neg\forall \mathbf{x} : \mathbf{A}.$$

Constructive Logic (Cont.)

- Proof of $\mathbf{A} \vee \mathbf{B} \leftrightarrow \neg(\neg\mathbf{A} \wedge \neg\mathbf{B})$:
  - We show intuitionistically $\neg(\mathbf{A} \vee \mathbf{B}) \leftrightarrow (\neg\mathbf{A} \wedge \neg\mathbf{B})$:
    - Assume $\neg(A \vee B)$. If A then $A \vee B$, a contradiction
      Similarly we get $\neg B$, therefore $\neg A \wedge \neg B$.
    - Assume $\neg A \wedge \neg B$, show $\neg(A \vee B)$.
      Assume $A \vee B$. If A then a contradiction with $\neg A$, s
  - Now it follows (classically):

$$(\mathbf{A} \vee \mathbf{B}) \leftrightarrow \neg\neg(\mathbf{A} \vee \mathbf{B}) \leftrightarrow \neg(\neg\mathbf{A} \wedge \neg$$

Constructive Logic (Cont.)

- **Weak disjunction and existential quantification** is
formulae $\neg(\neg A \land \neg B)$ and $\neg\forall x : A.\neg B$.
  - When using only weak disjunction, existential quantifica
atomic formulae, we obtain classical logic.

- **Strong disjunction and existential quantificaton** is
original type theoretic formulae.

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(f) The Set of Natural Numbers

- The set N is the type theoretic representation of the set $\mathbb{N}$
- $\mathbb{N}$ can be generated by
  - starting with the empty set,
  - adding 0 to it, and
  - adding, whenever we have x in it $x + 1$ to it.

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The Set of Natural Numbers (Cont.)

- Let S be a type theoretic notation for the operation $x \mapsto$
- Then the type theoretic rules are

$$\mathrm{N} : \mathrm{Set}$$

$$0 : \mathrm{N}$$

$$\frac{n : \mathrm{N}}{\mathrm{S}\ n : \mathrm{N}}$$

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Primitive Recursion

- **Primitive Recursion expresses:**
  Assume we have
  - $a : \mathrm{N}$.
  - and, if $n : \mathrm{N}$, $x : \mathrm{N}$ then $g\ n\ x : \mathrm{N}$.

  Then we can define $\mathrm{f} : \mathbf{N} \rightarrow \mathbf{N}$, s.t.
  - $f\ 0 = a$,
  - $f\ (\mathrm{S}\ n) = g\ n\ (f\ n)$.

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Primitive Recursion (Cont.)

- The **computation of $\mathbf{f}$ $\mathbf{n}$** proceeds now as follows:
  - Compute n.
  - If $n = 0$, then the result is a.
  - Otherwise $n = \mathrm{S}(n')$.
    - ∗ We assume that we have determined already how to
    - ∗ Now $f\ n$ reduces to $g\ n'\ (f\ n')$.
    - ∗ $g\ n'\ (f\ n')$ can be computed, since we know how to
      - · g
      - · $f\ n'$.

Example

- The function $f : \mathrm{N} \to \mathrm{N}$ with $f(x) = 2 \cdot x$ can be **recursively** by:
  - $f(0) = 0$.
  - $f(\mathrm{S}\ n) = \mathrm{S}\ (\mathrm{S}\ (f\ n))$.
- Therefore take in the definition above:
  - $a = 0$,
  - $g\ n\ x = \mathrm{S}\ (\mathrm{S}\ x)$.

Generalized Primitive Recursion

- We can **generalize primitive recursion** as follows:
  - First we can **replace the range of f by an arbitrary s**
    - i.e. we allow for any set C

$$f : \mathrm{N} \to C$$

  - Further, C can now **depend on N**.
- We obtain the following set of rules:

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Rules for the Natural Numbers

Formation Rule

$$\mathrm{N} : \mathrm{Set}$$

Introduction Rules

$$0 : \mathrm{N}$$

$$\frac{n : \mathrm{N}}{\mathrm{S}\ n : \mathrm{N}}$$

Elimination Rule

$$\frac{\begin{array}{c} C : \mathrm{N} \to \mathrm{Set} \\ a : C\ 0 \\ f : (x : \mathrm{N}) \to C\ x \to C\ (\mathrm{S}\ x) \\ n : \mathrm{N} \end{array}}{\mathrm{P}\ C\ a\ f\ n : C\ n}$$

Equality Rules

$$\mathrm{P}\ C\ a\ f\ 0 = a$$

$$\mathrm{P}\ C\ a\ f\ (\mathrm{S}\ n) = f\ n\ (\mathrm{P}\ C\ a\ f\ n)$$

Rules for the Natural Numbers (Cont.)

- Note that if we define in the elimination rule $g := \mathrm{P}\ C\ f$
  - The conclusion of the elimination rule reads:

$$g\ n : C\ n$$

which means that

$$\lambda(n : \mathrm{N}).g\ n : (n : \mathrm{N}) \to C\ n\ \ .$$

  - The equality rules read:

$$\begin{array}{rcl} g\ 0 & = & a \\ g\ (\mathrm{S}\ n) & = & f\ n\ (g\ n) \end{array}$$

Rules for N using the Logical Framework

- The more compact notation is:
  - $\mathrm{N} : \mathrm{Set}$,
  - $0 : \mathrm{N}$,
  - $S : \mathrm{N} \to \mathrm{N}$,
  - $\mathrm{P} \ : (C : \mathrm{N} \to \mathrm{Set}) \to C\ 0 \to ((x : \mathrm{N}) \to C\ x \to C\ (S\ x)) \to (n : \mathrm{N}) \to C\ n$.

Natural Numbers in Agda

- N is defined using **data**:

$$\mathrm{data\ N} = \mathrm{Z} \mid \mathrm{S}(n :: N)$$

(Unfortunately, 0 is not an acceptable name in Agda).

- Therefore we have

$$\begin{array}{lll} \mathrm{Z} & :: & \mathrm{N} \\ \mathrm{S} & :: & \mathrm{N} \rightarrow \mathrm{N} \end{array}$$

Elimination Rules for N in Agda

- Elimination works via case distinction in Agda.
  - If we want to introduce

$$\begin{array}{ll} f & (n :: \mathrm{N}) \\ & :: \quad A \\ & = \quad \{! \quad !\} \end{array}$$

    * A possibly depending on n,

    we can type into the goal n and use the menu agda-cas
    We get

$$\begin{array}{ll} f & (n :: \mathrm{N}) \\ :: & A \\ = & \text{case } n \text{ of} \\ & \quad \begin{array}{llll} \{ & (\mathrm{Z}) & \rightarrow & \{! \quad !\}; \\ & (\mathrm{S}\ n') & \rightarrow & \{! \quad !\}; \} \end{array} \end{array}$$

Elimination Rules for N in Agda (Cont

- For solving the goals, we can now **make use of f**.
  That will be **accepted by the type checker.**

- However, if we use of full f, and then use menu item **"agda-check-termination"**, we **might obtain an error-**

- If we

  – **do not make use of f in the case n=Z** and
  – **only use of $\mathrm{f}\ \mathrm{n}'$ in case n $= \mathrm{S}\ \mathrm{n}'$**.

  then **agda-check-termination succeeds**.

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Elimination Rules for N in Agda (Cont

- If **agda-check-termination succeeds**, the definition shou
  - (The lecturer hasn't checked the algorithm).
- However, **if agda-check-termination fails**, the **definiti correct**.

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Example of the Power of Termination Ch

- The following definition of the **Fibonacci numbers** can't
directly using the rules of type theory, but it **can be de**
follows and **agda-check-termination accepts it**:
(one := S Z):

$$
\begin{array}{lll}
\mathrm{fib} & (n :: N) & \\
 & :: & \mathrm{N} \\
 & = & \mathrm{case}\ n\ \mathrm{of} \\
 & & \quad \{\ (\mathrm{Z}) \rightarrow \mathrm{one}; \\
 & & \quad\ \ (\mathrm{S}\ n') \rightarrow \mathrm{case}\ n'\ \mathrm{of} \\
 & & \qquad \{\ (\mathrm{Z}) \rightarrow \mathrm{one}; \\
 & & \qquad\ \ (\mathrm{S}\ n'') \rightarrow \mathrm{fib}\ n'
\end{array}
$$

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Example for Limitations of Termination C

- Assume we define the **predecessor function**

$$\begin{array}{lll} \mathrm{pred} & (n :: \mathrm{N}) & \\ & :: & \mathrm{N} \\ & = & \mathrm{case}\ n\ \mathrm{of} \\ & & \quad \{\ (\mathrm{Z}) \quad \rightarrow \quad \mathrm{Z}; \\ & & \quad\ \ (\mathrm{S}\ n') \quad \rightarrow \quad n';\} \end{array}$$

i.e

$$\mathrm{pred}(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise.} \end{cases}$$

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Example for Limitations of Termination Check

- Then the function

$$\begin{array}{lll} f & (n :: \mathrm{N}) & \\ & :: & \mathrm{N} \\ & = & \mathrm{case}\ n\ \mathrm{of} \\ & & \quad \{\ (\mathrm{Z}) \quad \rightarrow \quad \mathrm{Z}; \\ & & \quad \quad (\mathrm{S}\ n') \quad \rightarrow \quad f\ (\mathrm{pred}\ n); \} \end{array}$$

  terminates always

  – (it returns for all $n : \mathrm{N}$ the value Z).

- However, **agda-check-termination fails**.

Limitations of the Termination Check (Co

- Because of the **undecidability of the Turing halting pro**
  - it is undecidable whether a recursively defined function
- there is no **extension of agda-check-termination, whic all in agda definable functions, which terminate for a**

Example: Addition

- Definition of $+$ in Agda:

$$\begin{array}{lll} (+) & (n, m :: \mathrm{N}) & \\ & :: & \mathrm{N} \\ & = & \mathrm{case}\ m\ \mathrm{of} \\ & & \quad \{\ (\mathrm{Z}) \quad \rightarrow \quad n; \\ & & \quad\ \ (\mathrm{S}\ m') \quad \rightarrow \quad \mathrm{S}\ (n + m');\} \end{array}$$

- The definition expresses:

$$\begin{array}{rcl} n + 0 & = & n \\ n + (m' + 1) & = & (n + m') + 1 \end{array}$$

Example: Addition

- Note that $(+)$ is used **infix**, i.e. we write $n+m$ for $(+)\ n$
- If $m = Sm'$, the definition of $(+)\ n\ m$ refers to $(+)\ n\ m'$
  - $(+)\ n\ m'$ is **defined before** $(+)\ \mathbf{n}\ \mathbf{m}$ since $\mathbf{m}'$ **is intro**

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Example: Multiplication

- Definition

$$
\begin{array}{lll}
(*) & (n, m :: \mathrm{N}) & \\
 & :: & \mathrm{N} \\
 & = & \text{case } m \text{ of} \\
 & & \quad \{\ (\mathrm{Z}) \quad \rightarrow \quad \mathrm{Z}; \\
 & & \quad \ \ (\mathrm{S}\ m') \quad \rightarrow \quad n * m' + n;
\end{array}
$$

- The definition expresses:

$$
\begin{array}{rcl}
n \cdot 0 & = & 0 \\
n \cdot (m' + 1) & = & (n \cdot m') + n
\end{array}
$$

Example: Multiplication (Cont.)

- Again $*$ is **treated infix**.
- Agda has built in that $*$ **binds more than** $+$.
  - $n * m' + n$ is treated as $(n * m') + n$.
- Note that the definition of $*$ requires, that $+$ **is already**

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Equality on N

- The **equality (n == m) ::Set** for $n, m :: \mathrm{N}$ can be equations:
  - $(\mathrm{Z} == \mathrm{Z}) = \mathrm{True}.$
  - $(\mathrm{Z} == \mathrm{S}\ n) = (\mathrm{S}\ n == \mathrm{Z}) = \mathrm{False}.$
  - $(\mathrm{S}\ n == \mathrm{S}\ m) = (n == m).$

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Equality on N (Cont.)

- From this one can now derive a definition in Agda:

$$
\begin{array}{lll}
(==) & (n, m :: \mathrm{N}) & \\
 & :: \quad \mathrm{Set} & \\
 & = \quad \mathrm{case}\ n\ \mathrm{of} & \\
 & \qquad \{ \ (\mathrm{Z}) \quad \rightarrow \quad \mathrm{case}\ m\ \mathrm{of} & \\
 & \qquad\qquad\qquad \{ \ (\mathrm{Z}) \quad \rightarrow & \\
 & \qquad\qquad\qquad\quad (\mathrm{S}\ m') \quad \rightarrow & \\
 & \qquad\quad (\mathrm{S}\ n') \quad \rightarrow \quad \mathrm{case}\ m\ \mathrm{of} & \\
 & \qquad\qquad\qquad \{ \ (\mathrm{Z}) \quad \rightarrow & \\
 & \qquad\qquad\qquad\quad (\mathrm{S}\ m') \quad \rightarrow &
\end{array}
$$

- Task of coursework 3, Question 1 to fill in those goals.

Reflexivity of ==

- **Reflexivity** of $==$ is the formula:

$$\forall n : \mathrm{N}.n == n$$

- **Type theoretically** this means that we have to define a f

$$\begin{array}{lll} \mathrm{refl} & (n : \mathrm{N}) & \\ & :: & n == n \\ & = & \{! \quad !\} \end{array}$$

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Reflexivity of == (Cont.)

- This can now be shown using **case distinction**:

$$
\begin{array}{lll}
\mathrm{refl} & (n : \mathrm{N}) & \\
 & :: & n == n \\
 & = & \mathrm{case}\ n\ \mathrm{of} \\
 & & \quad \{\ (\mathrm{Z}) \quad \rightarrow \quad \{!\ \ !\}; \\
 & & \quad\ \ \ (\mathrm{S}\ n') \quad \rightarrow \quad \{!\ \ !\};\}
\end{array}
$$

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Reflexivity of == (Cont.)

- Case $n = \mathrm{Z}$ is trivial.
- Case $n = \mathrm{S}\ n'$ can be solved using $\mathrm{refl}\ n'$ (which is define
- Task of Coursework 3, Question 1 (e) to solve this goal.

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Symmetry of ==

- **Symmetry** of $==$ is the formula:

$$\forall n, m : \mathrm{N}. n == m \rightarrow m == n$$

- **Type theoretically** this means that we have to define a f

$$\begin{array}{ll} \mathrm{sym} & (n, m : \mathrm{N}) \\ & (p :: n == m) \\ & :: \quad m == n \\ & = \quad \{! \quad !\} \end{array}$$

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Symmetry of == (Cont.)

- This can now be shown using **case distinction**:

$$
\begin{array}{lll}
\text{sym} & (n, m : \mathrm{N}) & \\
 & (p :: n == m) & \\
 & :: \quad m == n & \\
 & = \quad \text{case } n \text{ of} & \\
 & \quad \{ \ (\mathrm{Z}) \quad \rightarrow \quad \text{case } m \text{ of} & \\
 & \qquad\qquad \{ \ (\mathrm{Z}) \quad \rightarrow \quad \{ & \\
 & \qquad\qquad \ \ (\mathrm{S}\ m') \quad \rightarrow \quad \{ & \\
 & \quad\ \ (\mathrm{S}\ n') \quad \rightarrow \quad \text{case } m \text{ of} & \\
 & \qquad\qquad \{ \ (\mathrm{Z}) \quad \rightarrow \quad \{ & \\
 & \qquad\qquad \ \ (\mathrm{S}\ m') \quad \rightarrow \quad \{ &
\end{array}
$$

Symmetry of == (Cont.)

- The **first goal** can be solved by using true (since $(\mathrm{Z} ==$
- For the **second goal** we know p is an element of $\mathrm{Z} == \mathrm{S}$
  - Therefore if we make **case distinction on p** we get

$$\mathrm{case}\ p\ \ \mathrm{of}\ \{\ \ \}$$

    and have solved the second goal.

- Similarly the **third goal can be solved**.

Symmetry of == (Cont.)

- In the fourth goal, we have as type of goal $\mathbf{S\ m' == S\ n}$
  to $\mathbf{m' == n'}$.
  - The type of p is $\mathrm{S}\ n' == \mathrm{S}\ m'$ which is identical to n'
- The goal can be solved by using $\mathrm{sym}\ n'\ m'\ p$.
  - Note that we can **use here p** since it is of **type $\mathbf{n' ==}$**
    * It is correct to use it since **$\mathbf{n'}$ is introduced before**
      · Therefore **$\mathbf{sym\ n'}$ can be defined before $\mathbf{sym\ n}$**.
    * This definition will be **accepted by agda-check-ter**

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Example: Tuples (or Vectors) of Length

- Define first

$$\begin{array}{rcl} \text{data Nil} & = & \text{nil} \\ \\ \text{Cons} & (A, B :: \text{Set}) \\ & :: & \text{Set} \\ & = & \text{data cons}(a :: A)(b :: B) \end{array}$$

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Example: Tuples (or Vectors) of Length

- Now we can define (we use Vec for vector)

$$
\begin{array}{lll}
\mathrm{Vec} & (A :: \mathrm{Set}) & \\
 & (n :: N) & \\
 & :: \quad \mathrm{Set} & \\
 & = \quad \mathrm{case}\ n\ \mathrm{of} & \\
 & \qquad \{\ (\mathrm{Z}) & \rightarrow \quad \mathrm{Nil}; \\
 & \qquad\ \ (\mathrm{S}\ m') & \rightarrow \quad \mathrm{Cons}\ A\ (\mathrm{Vec}\ A
\end{array}
$$

Tuples of Length n

- Therefore (with the obvious definition of two),

$$\mathrm{Vec}\ A\ n = \underbrace{\mathrm{Cons}\ A\ (\mathrm{Cons}\ A\ \cdots (\mathrm{Cons}\ A}_{n\ \text{times}}\,\mathrm{Nil})$$

- The elements of $\mathrm{Vec}\ A\ n$ are

$$\mathrm{cons}\ a_1\ (\mathrm{cons}\ a_2\ \cdots (\mathrm{cons}\ a_n\ \mathrm{nil}) \cdots)$$

  for elements $a_1, \ldots, a_n$ of A.

- In ordinary mathematical notation, we would write $\langle a_1, .$
  element.

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Remarks on Tuples of Length n

- In **ordinary mathematics**, we would define

$$
\begin{aligned}
\mathrm{Vec}(A,0) &:= \{\langle\rangle\} \ , \\
\mathrm{Vec}(A,n+1) &:= \{\langle a_1,\ldots,a_{n+1}\rangle \quad | \ a_1,\ldots,a_n
\end{aligned}
$$

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Remarks on Tuples of Length n

- If we define

$$\begin{aligned} \mathrm{nil} &:= \langle\rangle \ , \\ \mathrm{cons}(a_1, \langle a_2, \ldots, a_{n+1}\rangle) &:= \langle a_1, \ldots, a_{n+ \end{aligned}$$

then this reads:

$$\begin{aligned} \mathrm{Vec}(A, 0) &:= \{\mathrm{nil}\} \ , \\ \mathrm{Vec}(A, n+1) &:= \{\mathrm{cons}(a, b) \quad \mid a \in A \wedge b \in \mathrm{Ve} \end{aligned}$$

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Remarks on Tuples of Length n (Cont.

- In the type theoretic definition we have **constructors**
  - $\mathrm{nil} :: \mathrm{Vec}\ A\ \mathrm{Z}$
  - $\mathrm{cons@(Vec}\ A\ (\mathrm{S}\ n)) :: A \to \mathrm{Vec}\ A\ n \to \mathrm{Vec}\ A\ (\mathrm{S}\ n)$.
- This is the **type theoretic analogue** of the previous defin

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Example: Sum of Tuples of Length n

- Define

$$\begin{array}{lll} \mathrm{NVec} & (n :: N) & \\ & :: & \mathrm{Set} \\ & = & \mathrm{Vec}\ \mathrm{N}\ n \end{array}$$

- $\mathrm{Nvec}\ n$ are tuples of natural numbers of length n.

Example: Componentwise Sum of Tuples of L

- We define **component-wise sum of tuples of length n**.
  - Using mathematical notation, this sum for instance as t

$$\langle 2, 3, 4\rangle + \langle 5, 6, 7\rangle = \langle 7, 9, 11\rangle \ .$$

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Example: Componentwise Sum of Tuples of Leng

$$
\begin{array}{lll}
\text{SumNVec} & (n :: \text{N}) & \\
 & (avec, bvec :: \text{NVec}\ n) & \\
 & :: \quad \text{Nvec}\ n & \\
 & = \quad \text{case}\ n\ \text{of} & \\
 & \qquad \{\ (\text{Z}) \quad \rightarrow \quad \text{nil}; & \\
 & \qquad\ \ \ (\text{S}\ n') \quad \rightarrow & \\
 & \qquad\quad \text{case}\ avec\ \text{of} & \\
 & \qquad\qquad \{\ (\text{cons}\ a\ avec') \quad \rightarrow & \\
 & \qquad\qquad\quad \text{case}\ bvec\ \text{of} & \\
 & \qquad\qquad\qquad \{\ (\text{cons}\ b\ bvec') \quad - & \\
 & \qquad\qquad\qquad\quad \text{cons@_} & \\
 & \qquad\qquad\qquad\quad (a + b) & \\
 & \qquad\qquad\qquad\quad (\text{SumNVec}\ n'\ a &
\end{array}
$$

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(g) Lists

- We define the set of lists of elements of type `A` in Agda.
- We have two constructors:
  - nil, generating the empty list.
  - cons, adding an element of `A` in front of a list
- So we define lists as:

$$\begin{array}{lll} \mathrm{list} & (A :: \mathrm{Set}) & \\ & :: & \mathrm{Set} \\ & = & \mathrm{data}\ \mathrm{nil} \\ & & \qquad \mid \mathrm{cons}(a :: A)\ (l :: \mathrm{list}\ A) \end{array}$$

Elimination Rule for Lists

- Elimination rule uses list-recursion:
  Assume
  - $A : \mathrm{Set}$
  - $C :: \mathrm{Set}$, depending on $l :: \mathrm{list}\ A$.

  Then we can define

$$\begin{array}{llllll} f & (l :: \mathrm{list}\ A) & & & & \\ & :: & C & & & \\ & = & \mathrm{case}\ l\ \mathrm{of} & & & \\ & & & \{\ (\mathrm{nil}) & \rightarrow & \{!\ \ !\}; \\ & & & \ \ (\mathrm{cons}\ a\ l') & \rightarrow & \{!\ \ !\};\} \end{array}$$

  and in the second goal we can make use of $f\ l'$.

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Example: Length of a List

$$
\begin{array}{lll}
\text{length} & (l :: \text{list N}) & \\
 & :: & \text{N} \\
 & = & \text{case } l \text{ of} \\
 & & \quad \{ \; (\text{nil}) \quad \rightarrow \quad \text{Z}; \\
 & & \quad\;\;\; (\text{cons } a\ l') \quad \rightarrow \quad \text{S (length } \ldots
\end{array}
$$

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Example: Sum of the Elements of a Li

$$
\begin{array}{lll}
\text{sumlist} & (l :: \text{list N}) & \\
 & :: & \text{N} \\
 & = & \text{case } l \text{ of} \\
 & & \quad \{ \ (\text{nil}) \quad \longrightarrow \quad \text{Z}; \\
 & & \quad\ \ \ (\text{cons } n\ l') \quad \longrightarrow \quad n + \text{sumlis}
\end{array}
$$

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Interesting Exercise

- Define
$$\text{append} : (A : \text{Set}) \to (\text{list } A) \to (\text{list } A) \to \text{l}$$
s.t. $\text{append } A\ l\ l'$ is the result of appending the list l' at t

- E.g., if $\text{a}, \text{b}, \text{c}, \text{d}$ are elements of A,
and if we define $\text{cons} := \text{cons}@(\text{list } A)$, $\text{nil} := \text{nil}@(\text{list } A$
$$\text{append } A\ (\text{cons a}\ (\text{cons b nil}))\ (\text{cons c}\ (\text{cons}$$
$$= \text{cons a}\ (\text{cons b}\ (\text{cons c}\ (\text{cons d nil})))$$

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(h) Universes.

- A universe U is a set, the elements of which are codes for

- So we have
  - $\mathrm{U} : \mathrm{Set}$,
  - $\mathrm{T} : \mathrm{U} \rightarrow \mathrm{Set}$ (the decoding function).

- We consider in the following a universe closed under
  - Fin_0, Fin_1, Bool,
  - N,
  - $+$,
  - Σ,
  - the dependent function type.

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Rules for the Universe

Formation Rule

$$\mathrm{U} : \mathrm{Set}$$

$$\frac{a : \mathrm{U}}{\mathrm{T}\ a : \mathrm{Set}}$$

Introduction and Equality Rules

$$\widehat{\mathrm{Fin}_0} : \mathrm{U} \qquad \mathrm{T}(\widehat{\mathrm{Fin}_0}) = \mathrm{Fin}_0 : \mathrm{Set}$$

$$\widehat{\mathrm{Fin}_1} : \mathrm{U} \qquad \mathrm{T}(\widehat{\mathrm{Fin}_1}) = \mathrm{Fin}_1 : \mathrm{Set}$$

$$\widehat{\mathrm{Bool}} : \mathrm{U} \qquad \mathrm{T}(\widehat{\mathrm{Bool}}) = \mathrm{Bool} : \mathrm{Set}$$

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Introduction/Equality Rules for the Universe (Cont.)

$$\frac{a : \mathrm{U} \qquad b : \mathrm{U}}{a \mathbin{\widehat{+}} b : \mathrm{U}}$$

$$\mathrm{T}(a \mathbin{\widehat{+}} b) = \mathrm{T}(a) + \mathrm{T}(b) : \mathrm{Set}$$

$$\frac{a : \mathrm{U} \qquad b : \mathrm{T}(a) \rightarrow \mathrm{U}}{\widehat{\Sigma}(a, b) : \mathrm{U}}$$

$$\mathrm{T}(\widehat{\Sigma}(a, b)) = \Sigma\ \mathrm{T}(a)\ (\lambda x.\mathrm{T}\ (b\ x)) : \mathrm{Set}$$

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Introduction/Equality Rules for the Universe (Cont.)

$$\frac{a : \mathrm{U} \qquad b : \mathrm{T}(a) \to \mathrm{U}}{\widehat{\Pi}(a, b) : \mathrm{U}}$$

$$\mathrm{T}(\widehat{\Pi}(a, b)) = (x : \mathrm{T}(a)) \to \mathrm{T}\ (b\ x) : \mathrm{Set}$$

Elimination and Equality Rules for the Universe (Cont.)

- There exist as well elimination rules and corresponding ec
universe.
- They are very long (one step for each of constructor of U
much used.
- They follow the principles present in previous rules.

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Applications of the Universe

- Ordinary elimination rules don't allow to eliminate into Se
- However often, one can verify, that all sets needed a universe",
  - i.e. there are codes in the universe representing them.
- Then one can eliminate into the universe instead of Set a the required function.

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Applications of the Universe

- Example: Define

$$
\begin{array}{lcl}
\widehat{\text{atom}} & : & \text{Bool} \to \text{U} \quad , \\
\widehat{\text{atom}} & := & \text{Case}_{\text{Bool}}\ (\lambda(x : \text{Bool}).\text{U})\ \widehat{\text{Fin}_1}\ \widehat{\text{F}
\end{array}
$$

$$
\begin{array}{lcl}
\text{atom} & : & \text{Bool} \to \text{Set} \quad , \\
\text{atom} & : & \lambda(x : \text{Bool}).\text{T}\ (\widehat{\text{atom}}\ x) \quad ,
\end{array}
$$

Then

- $\text{atom tt} = \text{Fin}_1$,
- $\text{atom ff} = \text{Fin}_0$.

Universes in Agda

- U and T need to be defined simultaneously.
  - Usually Agda type checks definitions in sequence, so no
    definitions possible.
  - Special construct `mutual`.
    * Everything in the scope of it is type checked simultar
    * Scope determined by indentation.

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Universes in Agda (Cont.)

$$
\begin{array}{lll}
\text{mutual} & & \\
\quad \text{U} & :: & \text{Set} \\
 & = & \text{data Nhat} \\
 & & \qquad \mid \text{Finzerohat} \\
 & & \qquad \mid \text{Finonehat} \\
 & & \qquad \mid \text{Boolhat} \\
 & & \qquad \mid \text{Sigmahat } (a :: \text{U})(b :: \text{T } a \rightarrow \\
 & & \qquad \mid \text{Pihat } (a :: \text{U})(b :: \text{T } a \rightarrow \text{U})
\end{array}
$$

Universes in Agda (Cont.)

T in the following is to be intended the same as U:

$$
\begin{array}{lll}
\mathrm{T} & (u :: \mathrm{U}) & \\
& :: & \mathrm{Set} \\
& = & \text{case } u \text{ of} \\
& & \quad \{ \begin{array}{lll}
(\mathrm{Nhat}) & \rightarrow & N; \\
(\mathrm{Finzerohat}) & \rightarrow & \mathrm{Finzero}; \\
(\mathrm{Finonehat}) & \rightarrow & \mathrm{Finone}; \\
(\mathrm{Boolhat}) & \rightarrow & \mathrm{Bool}; \\
(\mathrm{Sigmahat}\ a\ b) & \rightarrow & \mathrm{Sigma}\ (\mathrm{T}\ a)\ (\lambda(x :: \\
(\mathrm{Pihat}\ a\ b) & \rightarrow & (x :: \mathrm{T}\ a) \rightarrow \mathrm{T}\ (b\ x)
\end{array}
\end{array}
$$

(i) Algebraic Data Types.

- The construct "data" in Agda is much more powerful th
by type theoretic rules.

- In general we can define now sets having arbitrarily many
arbitrarily many arguments of arbitrary types.

$$
\begin{array}{lll}
\mathrm{A} & :: & \mathrm{Set} \\
 & = & \mathrm{data}\ \mathrm{C}_1(a_{11} :: \mathrm{A}_{11}) \cdots (\mathrm{a}_{1\mathrm{n}_1} :: \mathrm{A}_{1\mathrm{n}_1}) \\
 & & \quad \mid \mathrm{C}_2(a_{21} :: \mathrm{A}_{21}) \cdots (a_{2n_2} :: \mathrm{A}_{2\mathrm{n}_2} \\
 & & \quad \cdots \\
 & & \quad \mid \mathrm{C}_\mathrm{m}(a_{m1} :: \mathrm{A}_{\mathrm{m}1}) \cdots (a_{mn_m} :: \mathrm{A}
\end{array}
$$

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Meaning of "data"

- The idea is that A as before is the least set A s.t. we have

```
Ci@A ::  (ai1 :: Ai1)
         -> ...
         -> (aini :: Aini)
         -> A
```

where a constructor always constructs new elements.

- In other words the elements of A are exactly those co
constructors.

Strictly Positive Algebraic Data Types

- In the types A_{ij} we can make use of A.
  - However, it is difficult to understand A, if we have **ne**
    of A.
  - Example:
    ```
    A ::  Set
      = data C (f ::  A -> A)
    ```
  - What is the least set A having a constructor
    ```
    C@A ::  (f ::  A -> A)
            -> A  ?
    ```

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Strictly Positive Algebraic Data Types (Co

– If we
  * have constructed some part of A already,
  * find a function f:: A -> A, and
  * add C@_ f to A,

  then f might no longer be a function A -> A.
  (f applied to the new element C@_ f might not be defi
– In fact, "agda-check-termination" issues a warning, if we

– We shouldn't make use of such definitions.

Strictly Positive Algebraic Data Types (Co

- A "good" definition is the set of lists of natural numbers,

```
Nlist ::  Set
      = data nil
           | cons (a::  N)
                  (l ::  Nlist)
```

- The constructor cons@_ of N-lists refers to Nlist, but in
  We have: if a :: N and l :: Nlist, then we hav
  Nlist.
  - If we add cons@_ a l to Nlist, the reason for addi
    Nlist) is not destroyed by this addition.
  - So we can "construct" the set Nlist by
    - starting with the emptyset,
    - adding nil@_ and
    - closing it under cons@_ whenever possible.

- Because we can "construct" Nlist, the above is an accep

Strictly Positive Algebraic Data Types (Co

- In general:

```
A :: Set
  = data C_1 (a_11:: A_11) ··· (a_1n_1:: A_1n_1)
         | C_2 (a_21:: A_21) ··· (a_2n_2:: A_2n_2)
         ···
         | C_m (a_m1:: A_m1) ··· (a_mn_m:: A_mn_m)
```

  is a strictly positive algebraic data type, if all A_{ij} are
  - either types which don't make use of A
  - or are A itself.
- And if A is a strictly positive algebraic data type, then A i
- The definitions of finite sets, $\Sigma\ A\ B$, $A + B$ and N we algebraic data types.

One further Example

- The set of binary trees can be defined as follows:

```
Bintree :: Set
        = data leaf
               | branch (left :: Bintree)
                        (right :: Bintree)
```

- This is a strictly positive data type.

Strictly Positive Algebraic Data Types Extensions

- An often used extension is to define several sets simultane
- Example: the even and odd numbers:

```
mutual
  Even ::  Set
       = data Z | S (n::  Odd)

  Odd  ::  Set
       = data S (n::Even)
```

- In such examples the constructors refer strictly positive to to be defined simultaneously.

Strictly Positive Algebraic Data Types, Extensions

- We can even allow $\mathtt{A_{ij}} = \mathtt{B_1} -> \mathtt{A}$ or even $\mathtt{A_{ij}} = \mathtt{B_1} ->$
  where A is one of the types introduced simultaneously.

- Example (called "Kleene's O":

```
O::  Set
 = data leaf
       | succ (o::O)
       | lim (f ::  N -> O)
```

- The last definition is unproblematic, since, if we have f
  construct lim@_ f out of it, adding this new element to
  the reason for adding it to O.

- So again O can be "constructed".

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Elimination Rules for data

- Functions from strictly positive data types can now b distinction as before.
- For termination we need only that in the definition of f, w `f (C@_ a`$_1$ `···` `a`$_n$`)`, we can refer only to f applied to ele `a`$_1$ `···` `a`$_n$.

- For instance
  - in the Bintree example, when defining

    $$\texttt{f} :: \texttt{Bintree}- > \texttt{A}$$

    by case-distinction, then the definition of

    $$\texttt{f}\ (\texttt{branch@_ left right})$$

    can make use of `f left` and `f right`.
  - In the example of `O`, when defining

    $$\texttt{g} :: \texttt{O}- > \texttt{A}$$

    by case-distinction, then the definition of

    $$\texttt{g}\ (\texttt{lim@_ f})$$

    can make use of `g (f n)` for all `n::  N`.