1. From Simple to Dependent Types

(a) Approaches to Verified Software
(b) The Theorem Prover Agda
(c) Concept of a Type
(d) Dependent Types
(e) Dependent Types in Programming

Approaches to Verified Software

(ii) Prove that a solution for the problem exists. Extract a program from it.
   E.g. from a proof of the statement
   For every list there exists a sorted list having the same elements
   one can extract a program, which computes from a list a sorted list having the same elements.
   The correctness is guaranteed.
   Technology not yet far developed.
   Dr. Berger and Dr. Seisenberger are experts in this area of research.

(a) Approaches to Verified Software

We consider 4 principal approaches towards writing verified software.

(i) First a program is written. Then its correctness is verified.
   Most common approach, when formal methods are applied.
   Main advantage: Ordinary programming languages can be used.
   Disadvantage: all or most considerations of the programmers are lost.
   Requires advanced automated theorem proving technologies.
   Dr. Kullmann is an expert on the theorem proving techniques used there.

(iii) Programs written in a language which allows to state properties of the program.
Example: “This program sorts a list”. Properties should be verified when compiling the program.
   Advantages:
   Programmer is forced to think very clearly.
   Programs will be very well documented.
   The information about properties needed might guide the programmer.
   In some cases parts of the program can even be found automatically.
Approaches to Verified Software

Disadvantages of (iii):
- Requires new programming languages.
- Still essentially an area of research. However advanced tools exist already.
- Might be too difficult for ordinary programmers.

Effect:
Proving and programming will be the same.

(iv) Mixtures between (i), (iii).
- E.g. SPARK Ada.

In this lecture, we will follow the approach of (iii), based on dependent type theory.

(b) The Theorem Prover Agda

Implementations of Dependent Type Theory

- **NuPrl** (Cornell, USA), a technically very advanced system.
  - Uses so-called “extensional type theory”.
- **Coq** (INRIA, France), as well technically very advanced.
- **LEGO** (Edinburgh), about to be replaced.
  - Coq and LEGO use so-called “impredicative type theory”.
- **Epigram** (Nottingham, Durham). New system, not yet as far developed as the other systems.

The “**Alf-family**” (Gothenburg, Sweden) – has probably the clearest concepts.
- **Alf** (developed by Lena Magnusson, written in SML)
- **Half** (= Haskell Alf), developed by Thierry Coquand, Dan Synek.
  - Original goal was to rewrite Alf using Haskell.
- **CHalf**, C-version of Half developed by Dan Synek.
- **Agda** developed by Catarina Coquand.
- **Alfa**, a graphical user interface for Agda, developed by Thomas Hallgren.
- **Agda2** developed by Ulf Norell.
- **"Cayenne"** (Gothenburg, Sweden) is a dependently typed programming language (not intended as a theorem prover), with a syntax very similar to Agda.
Proofs in Agda

- In this module we will use Agda2.
- **Half, Agda** are written in **Haskell**.
- **Half** and **Agda** have an **Emacs/XEmacs mode**, which makes it quite convenient to develop proofs in it.
- In most theorem provers, one has to follow one or several goals, and derive proofs for them. This is close to the way, **proofs are carried out by hand**.

Proofs in Agda (Cont.)

- The Alf-family has a different approach of **successive refinement**.
  - One starts writing the proof code similarly to writing functional programs.
  - What cannot be done without machine assistance can be left open in the form of holes (**goals**).
  - Now one can successively, assisted by the system, fill in those goals.
  - Therefore proof/program development in the Alf family is very close to ordinary programming.

Thierry Coquand

- The main theoretician behind Agda.

Catarina Coquand

(c) Concept of a Type

Typed vs. untyped languages

- **Examples of typed languages:**
  Pascal, C, C++, Java, C#, Haskell, ML, ...
- **Examples of untyped languages:**
  Perl, Python, Visual Basic, Lisp, ...

Advantages/Disadvantages

- **Advantage of untyped languages:**
  Greater freedom in programming.
- **Advantages of typed languages:**
  - Many errors are avoided, especially when using operations defined somewhere else. To find such errors in untyped languages can be very difficult.
  - Types are very natural comments to programs, which express the basic functionality of a program. Typed languages enforce this kind of comments, and therefore better documentation.

Concept of a Type (Cont.)

- In order to guarantee correctness of software, we make use of a much more refined type system.
  - It will allow to specify any property of a program, which can be defined as a formula, as a type.

Need for Rich Type Structures

- In general programming, one wants a very rich type structure.
  - The richer the type structure, the more data types one can define, the more flexibility one has when writing programs, without loosing the advantages of types (preventing errors).
  - In this lecture we will mainly focus on theorem proving.
  - Greater flexibility touched at the end.
Types used in other Languages

- **Scalar types:**
  Booleans, integers, floating point numbers, characters, enumeration types.

- **Simple compound types:**
  Arrays, strings, record types, lists, sets.

- In **functional programming** additionally:
  Function types, algebraic types (= what can be defined using “data”).

- In **object-oriented programming** (not relevant here):
  interfaces (and classes).

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Types used in Dep. Type Theory

- **Function types.**
  Let \texttt{NatList} the type of lists of natural numbers.
  Then \texttt{NatList} $\rightarrow$ \texttt{NatList} is the type of functions mapping lists of natural numbers to lists of natural numbers.
  E.g. sorting functions (without any correctness conditions) are elements of this type.

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Types used in Dep. Type Theory

- **Products** (essentially records).
  \texttt{Int} $\times$ \texttt{Char} is the type of pairs \langle r, s \rangle, where \( r \) is an integer and \( s \) is a character.
  E.g. \langle 2, 'c' \rangle : \texttt{Int} $\times$ \texttt{Char}.
  In Haskell notation, products are written as follows:
  Haskell notation for \( A \times B \) is \( (A,B) \),
  Haskell notation for \langle a, b \rangle is \( (a,b) \).
  e.g. \( (2,3) :: \texttt{(Int,Int)} \).
  Agda notation will be that of a record type in other languages.

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Types used in Dep. Type Theory

- **Algebraic types.**
  Types denoted in Haskell by the keyword \texttt{data}.

- **Dependent versions** of the above.
(d) Dependent Types

Assume we want to assign a type to a sorting function sort on lists of natural numbers.

In most programming language, the type of it is essentially

\( \text{sort} : \text{NatList} \rightarrow \text{NatList} \)

for the type of lists of natural numbers \( \text{NatList} \).

In dependent type theory, we can demand more correctness, namely that its type is

\( \text{sort} : \text{NatList} \rightarrow \text{SortedList} \).

We assume some notion of \( \text{NatList} \) (list of natural numbers).

Sorted Lists

For the moment, ignore what is meant by \( \text{SortedList} \) as a type.

Only important: \( \text{SortedList} \) depends on \( \text{l} \).

- \( \text{SortedList} \) is a predicate expressed as a type.

Elements of SortedList are pairs \( \langle l, p \rangle \) s.t.
- \( l : \text{NatList} \).
- \( p : \text{Sorted} \).

Sorted \( l \) is a dependent type.

Sorted Lists (Cont.)

An element of \( \text{SortedList} \) will be a \textbf{proof} that \( l \) is sorted.

If \( l \) is sorted, then \( \text{SortedList} \) will be provable, and therefore \textbf{will have an element}.

- It is possible to write a program which computes an element of \( \text{SortedList} \).

If \( l \) is not sorted, then \( \text{SortedList} \) will have no proof and it will therefore \textbf{no element}.

Then it is not possible to write a program which computes an element of \( \text{SortedList} \).
The Dependent Product

- Then the pair \(⟨l, p⟩\) will be an element of
  \[
  \text{SortedList} := (l : \text{NatList}) \times (\text{Sorted } l).
  \]
- \text{SortedList} is the type of pairs \(⟨l, p⟩\) s.t.
  - \(l : \text{NatList},\)
  - \(p : \text{Sorted } l.\)
  called the **dependent product**
- \text{sort} : \text{NatList} → ((l : \text{NatList}) \times (\text{Sorted } l)) expresses:
  - \text{sort} converts lists into sorted lists.

The Dependent Function Type

- A refined version of \text{sort} has type
  \[
  (l : \text{NatList}) → ((l' : \text{NatList}) \times (\text{Sorted } l') \times (\text{EqElements } l \ l'))
  \]
- “\text{sort} l is a list, which is sorted and has the same elements”.
- “\text{sort} is a program, which takes a list and returns a sorted list with the same elements.”
- The type of \text{sort} is an instance of the **dependent function type**:
  - The result type depends on the arguments.

(e) Dependent Types in Programming

Dependent types are often needed in programming, even if no verification is needed. We give some examples:

- In Java, a relatively big library of "collection classes" is available.
  - Provides implementations of lists, sets, hash tables, etc.
  - It would be nice to have "lists of type \(A\)".
  - However this is a dependent type, depending on a type \(A\).
  - Cannot be expressed in old versions of Java (without templates).
  - Instead, in Java only lists of elements of type \text{Object} are available.
Dependent Types in Programming

Elements of other types have to be upcasted to Object. Elements of the list have then to be downcasted to their original type. Type checking happens at run time rather than at compile time.

Example

Assume a class StudentEntry.

If we have a list listOfStudentEntries, and add to it an element studentEntry of type StudentEntry, this element will first be converted (upcasted) to type Object.

If we retrieve an element (e.g. the first element) of listOfStudentEntries, we obtain an element of Object.

If it was originally a StudentEntry, we can cast this element down to StudentEntry.

However, whether we have an element of StudentEntry, cannot be determined at compile time, only at run time.

Polymorphism

What is needed is a weak form of dependent types, called polymorphism. Types might depend on other types but not on elements of types.

In C++, this form of dependency is available (called templates).

One writes for instance List<A> for lists of type A.

In Java they are available from Java 5 onwards (with major restrictions).

In C#, it is available from C# 2 onwards (as well with major restrictions).

In Haskell and ML, it is available.

E.g. \( x \rightarrow x :: \alpha \Rightarrow \alpha \rightarrow \alpha \), which means that \( \lambda x.x \) is of type \( \alpha \rightarrow \alpha \) for every type \( \alpha \).
Matrix Multiplication

- **Matrix multiplication** is an operation, which takes three natural numbers
  \[ n, m, k \]
  an \( n \times m \)-matrix and an \( m \times k \)-matrix, and has as result an \( n \times k \)-matrix.

The type of this function is a **dependent type**:

The types of \( n \times m \)-matrices, of \( m \times k \)-matrices and of \( n \times k \)-matrices depend on \( n, m, k \).

Type of Matrix Multiplication

- Let \( N \) be the type of natural numbers (i.e. \( 0, 1, \ldots; N \) will be introduced later).

- Let \( \text{Mat} \ n \ m \) be the type of \( n \times m \)-matrices. (Will be introduced later).
  Then matrix multiplication has type

\[
(n : N) \\
\rightarrow (m : N) \\
\rightarrow (k : N) \\
\rightarrow \text{Mat} \ n \ m \\
\rightarrow \text{Mat} \ m \ k \\
\rightarrow \text{Mat} \ n \ k
\]

Matrix Multiplication (Cont.)

- Usually, this problem is solved by
  - taking matrices which are big enough and restricting the operation to \( n \times m, m \times k \) and \( n \times k \) sub-matrices,
  - waste of memory
  - or by dynamically allocating arrays.
  - This means memory allocation has to be done at run time.
  - In both solutions, checking that the dimensions are in accordance has to be done at **run-time**.
Dependent Types in Programming

Digital Components.
- A digital component (e.g. a logic gate) with $n$ inputs and $m$ outputs can be considered as a function $\text{Bool}^n \rightarrow \text{Bool}^m$.
- This function type is a dependent type, since it depends on the parameters $n$ and $m$.

Predicates are dependent types.
- See the types of sort above.

Dependent Types in Linguistics

- Aarne Ranta has used dependent types in linguistics:
  - In a sentence like “The man goes home”, the verb (“goes”) depends on,
    - whether the noun (“the man”) is singular (then the predicate is “goes home”)
    - or plural (then the predicate is “go home”).
  - Aarne Ranta constructed grammars based on dependent types, and used them for translating sentences between different languages automatically.

More Advanced Applications

- Hongwei Xi and Robert Harper have introduced a dependently typed assembly language (DTAL), which allows to guarantee that array bounds are not violated, without having to carry out run time checks.
- Petri Mäenpää and Matti Tikkanen have used dependent types for the modelling of databases.
- Simon Thompson, Erik Poll and John Schackell have proposed the use of dependent types in order to integrate logic (based on dependent types) into computer algebra systems.
  (From O. Solaja: The verification of commonly used algorithms in dependent type theory, 3rd year project, Swansea 2004, p. 4.)