

## Proposed title of talk

A sub-system of Girard's  $\mathbf{F}$  with ordinal strength  $\varepsilon_0$

## Related work

It has been known since its inception that many classes of recursive function, for example the primitive recursive functionals (henceforth  $\mathbf{PRF}$ 's), are included within the class of functions defined by terms of  $\mathbf{F}$ . But until Leivant [2001], no-one had isolated a sub-system of  $\mathbf{F}$  in which the functions definable are *precisely* the  $\mathbf{PRF}$ 's. Leivant's treatment makes no use of ordinal numbers and is not therefore – at least on the face of it – ordinally informative. See also Altenkirch and Coquand [2001].

The name “Gödel's  $\mathbf{T}$ ” was given to theories traditionally used for talking about the  $\mathbf{PRF}$ 's; unlike  $\mathbf{F}$  they include what Curry et al. [1972, p.217] call “arithmetical” as well as “pure” combinators. One of the most important properties of  $\mathbf{T}$  is that a function can be defined by  $\varepsilon_0$ -recursion which takes any term to its normal form. Such facts as the fact that every  $\mathbf{PRF}$  is extensionally equal to a functional defined by  $\alpha$ -recursion (for some  $\alpha < \varepsilon_0$ ) follow from this key result. Tait [1965] and Howard [1980] are perhaps the two most important works in this area.

Until a similar result has been proved for a suitable fragment of  $\mathbf{F}$ , it will have to be said that  $\mathbf{F}$  lags somewhat behind  $\mathbf{T}$  in terms of its value as a precise tool for talking about the  $\mathbf{PRF}$ 's. The purpose of this lecture is to make good that defect.

The sub-system of  $\mathbf{F}$  that is treated is  $\mathbf{F}$  restricted to terms in which, if a subterm of the shape  $E\sigma F$  occurs, the type of  $E$  contains only one quantifier and that at the beginning (it is easy to prove that this property is preserved by beta- and type-reductions).  $\sigma$  by contrast may be as complicated as you like. That this fragment of  $\mathbf{F}$  is sufficient for  $\lambda$ -defining all  $\mathbf{PRF}$ 's follows from the fact that the Church numerals can be used as iterators upon terms of any type when they are assigned the type  $\forall R((R \rightarrow R) \rightarrow R \rightarrow R)$ .

## Importance of the problem

Looking at it in the broadest terms possible, the issue is whether (any sub-system of) the *pure*  $\lambda$ -calculus can be usefully used for studying the  $\mathbf{PRF}$ 's or any other proper sub-class of the general recursive functions. The goal of the present talk is just to show that one necessary condition for an affirmative answer is fulfilled. While it is convenient, it was not at all essential to concentrate on typed terms of Girard's  $\mathbf{F}$ : the same point could have been made with reference to some other typed  $\lambda$ -calculi or even to the untyped  $\lambda$ -calculus (and in fact the methods I am using were originally developed to analyse a theory that assigned types to untyped  $\lambda$ -terms).

## The basic idea of the proof

Following Tait [1965], Howard [1980] and many other articles, the basic idea is to think of the terms of the sub-system to be treated as being built up by rules *in the style of a sequent calculus* and take note of the fact that cut-free derivations always yield terms in normal form. Then a method of cut-elimination using transfinite induction is applied in order to show that any derivation of a term can be converted stepwise into a cut-free derivation of a reduct of that term.

In this case the method of cut-elimination was invented by Takeuti [1955]; Arai [1997; pp.6–9] has reproved Takeuti's result introducing valuable technical simplifications.

Especially in view of the fact that talks are likely to be limited to 30 minutes, it would seem more appropriate at this point to concentrate on the wood than on the trees, so I will just sketch the main ideas as clearly and simply as possible.

## References

- Altenkirch, T. and T. Coquand, 2001: 'A Finitary Subsystem of the Polymorphic Lambda Calculus' in S. Abramsky (ed.): *Typed Lambda Calculi and Applications* (Lecture Notes in Computer Science 2044) (Springer), pp.22–28.
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- Howard, W. A., 1980: 'Ordinal Analysis of Terms of Finite Type', *Journal of Symbolic Logic* 45, pp.493–504.
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