Unions of Reducibility Families

for λ-Calculus with Orthogonal Rewriting

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General Motivations

Termination of extensions of typed λ-calculus.

- Proofs assistants (strong normalization).
- Functional programming.

λ-Calculus with Rewriting

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- **Terms**

$$t, u \in \Lambda(\Sigma) \quad ::= \quad x \quad | \quad \lambda x.t \quad | \quad t\,u \quad | \quad \mathtt{f}(t_1, \ldots, t_n)\ ,$$

where $\mathtt{f} \in \Sigma_n$.

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- A type system (eg. simple types).

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- A type system (eg. simple types).

- **Rewrite Rules** of the form

$$\mathtt{f}(l_1, \dots, l_n) \mapsto_{\mathcal{R}} r\ ,$$

  where

$$\frac{\Gamma \vdash l_1 : T_1 \quad \dots \quad \Gamma \vdash l_n : T_n}{\Gamma \vdash \mathtt{f}(l_1, \dots, l_n) : T} \qquad \text{and} \qquad \Gamma \vdash r : T$$

Strong Normalization

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  No infinite sequence

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- Tools to prove that

$$\text{if} \quad \vdash t : T \quad \text{then} \quad t \in \mathcal{SN}$$

Type Interpretation

Let $\rightarrow_{\mathrm{R}}$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Interpretation of Types**

$$\mathrm{T} \in \mathcal{T} \quad \mapsto \quad [\![\mathrm{T}]\!] \subseteq \mathcal{SN}$$

- **Adequacy**

$$\vdash \mathrm{t} : \mathrm{T} \qquad \Longrightarrow \qquad \mathrm{t} \in [\![\mathrm{T}]\!]$$

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Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

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- $\mathcal{R}ed$ is a complete lattice

Different reducibility families:

- Tait's Saturated Sets [Tai75]
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Property used eg. in [BR06, Abe06, Tat07].

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- **Interpretation of types**

$$T \in \mathcal{T} \quad \mapsto \quad [\![T]\!] \in \mathcal{R}ed$$

- Sufficient conditions on $\mathcal{R}ed$ to get an **adequate** interpretation:

$$\vdash t : T \qquad \Longrightarrow \qquad t \in [\![T]\!]$$

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if $E[x] \to_\beta v$ then the reduction is in $E[\,]$,
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- **Weak Standardization**
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A β-reduct of $(\lambda x.t)u$ is either $t[u/x]$ or $(\lambda x.t')u'$ with $(t,u) \to_\beta (t',u')$.

- **Consequence:**
If $E[t[u/x]] \in \mathcal{SN}_\beta$ and $u \in \mathcal{SN}_\beta$ then $E[(\lambda x.t)u] \in \mathcal{SN}_\beta$.

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- $S \subseteq \mathcal{SN}_{\beta}$ is a **Saturated Set** ($S \in \mathcal{SAT}$) iff

($\mathcal{SAT}1$) if $E[\,] \in \mathcal{SN}_{\beta}$ and $x \in \mathcal{X}$ then $E[x] \in S$,
($\mathcal{SAT}2_{\beta}$) if $E[t[u/x]] \in S$ and $u \in \mathcal{SN}_{\beta}$ then $E[(\lambda x.t)u] \in S$.

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  stability by reduction (if $t \in \mathcal{S}$ and $t \to_{\beta\mathcal{R}} u$ then $u \in \mathcal{S}$),
  and that for all $E[\mathsf{f}(t_1, \dots, t_n)]$,

$$\begin{array}{ccccc} & & E[\mathsf{f}(t_1, \dots, t_n)] & & \\ & \swarrow_{\beta\mathcal{R}} & & {}_{\beta\mathcal{R}}\searrow & \\ u_1 \in \mathcal{S} & & \dots & & u_n \in \mathcal{S} \end{array} \quad \Longrightarrow \quad E[\mathsf{f}(t_1, \dots, t_n)] \in \mathcal{S}$$

Girard's Reducibility Candidates

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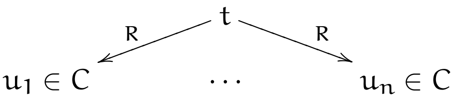

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$$t \xrightarrow{R} u_1 \in C \quad \cdots \quad t \xrightarrow{R} u_n \in C \quad \Longrightarrow \quad t \in C$$

$$E[u_1] \in \mathcal{SN} \quad \cdots \quad E[u_n] \in \mathcal{SN} \quad \Longrightarrow \quad E[t] \in \mathcal{SN}$$

Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

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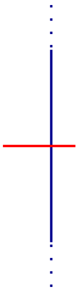

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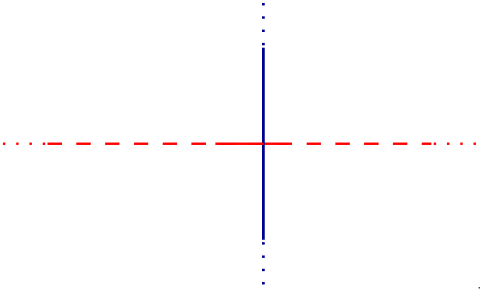

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Lemma

$$\emptyset \neq A \subseteq \mathcal{SN} \quad \Longrightarrow \quad A^{\perp\!\!\!\perp\perp\!\!\!\perp} \in \mathfrak{CR}$$

Outline

- Given a typed rewrite system $\mathcal{R}$,
- find a reducibility family $\mathfrak{Red}$
  which leads to an adequate type interpretation
  and such that

$$\emptyset \neq \mathfrak{R} \subseteq \mathfrak{Red} \quad \Longrightarrow \quad \bigcup \mathfrak{R} \in \mathfrak{Red}$$

Union Types

$$T_1, T_2 \in \mathcal{T} \quad ::= \quad \dots \quad | \quad T_1 \sqcup T_2$$

- We put

$$[\![T_1 \sqcup T_2]\!] \quad =_{\text{def}} \quad \mathcal{R}ed([\![T_1]\!] \cup [\![T_2]\!])$$

  This validates

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This validates
$$(\sqcup\text{I})\frac{\Gamma \vdash t : T_i}{\Gamma \vdash t : T_1 \sqcup T_2}$$
- If $\mathcal{R}ed$ stable by union, we have
$$[\![T_1 \sqcup T_2]\!] \quad = \quad [\![T_1]\!] \cup [\![T_2]\!]$$
This is sufficient to validate
$$(\sqcup\text{E})\frac{\Gamma \vdash t : T_1 \sqcup T_2 \qquad \begin{array}{c}\Gamma, x : T_1 \vdash c : C \\ \Gamma, x : T_2 \vdash c : C\end{array}}{\Gamma \vdash c[t/x] : C}$$

Unsafe Interaction [Rib07b]

$$t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_1 \qquad\qquad t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_2$$

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$$\dfrac{\dfrac{t_1 : T_1 \qquad t_2 : T_2}{t_1 + t_2 : T_1 \sqcup T_2} \qquad \begin{array}{c} x : T_1 \vdash xx : C \\ x : T_2 \vdash xx : C \end{array}}{}$$

Because

$$t_1 t_1 \in \mathcal{SN} \qquad \text{and} \qquad t_2 t_2 \in \mathcal{SN}$$

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Similar example with a **confluent** rewrite system.

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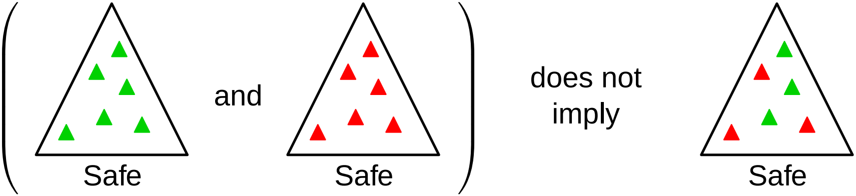

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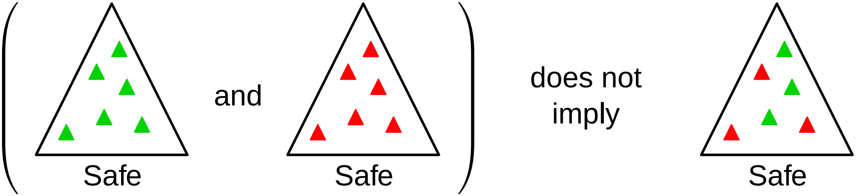

Prevents from having $[\![T_1 \sqcup T_2]\!] = [\![T_1]\!] \cup [\![T_2]\!]$.

Sufficient Conditions for $[\![T_1 \sqcup T_2]\!] = [\![T_1]\!] \cup [\![T_2]\!]$

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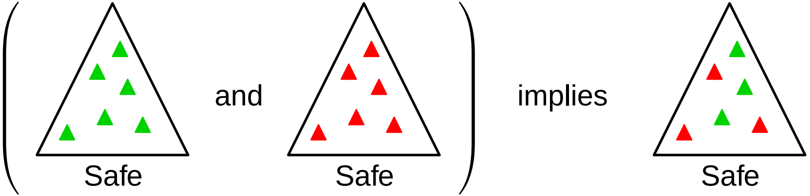

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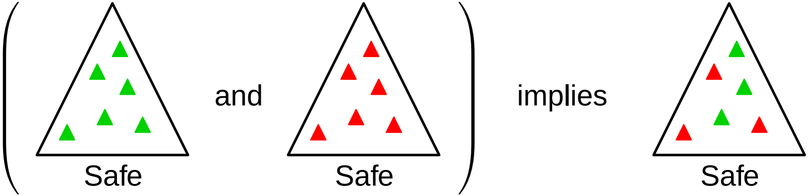
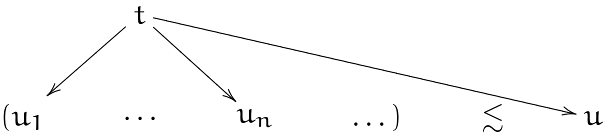


OK if  i.e. if t neutral has a "principal reduct" u:



Reducibility Candidates [Rib07a]

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If C is a reducibility candidate, then C is downward closed wrt. $\lesssim_{\mathfrak{CR}}$

- But not every such C is a reducibility candidate.

Stability by Union of Reducibility Candidates [Rib07a]

Theorem

The following are equivalent:

(i) $\mathfrak{CR}$ *is stable by union,*

(ii) $\mathfrak{CR}$ *is the set of all non-empty subsets* C *of* $\mathcal{SN}$ *that are downward closed wrt.* $\lesssim_{\mathfrak{CR}}$.

(iii) *for every* t *which is non-normal, strongly normalizing and neutral, there is a term* u *such that*

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- This holds for the λ-calculus with products and sums (also [Tat07]).

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Lemma

$\mathfrak{CR}$ *stable by union* $\Longrightarrow$ $\mathcal{O} \subseteq \mathfrak{CR}$

The converse is false, consider

$$p \mapsto_{\mathcal{R}} \lambda x.c_1 \qquad p \mapsto_{\mathcal{R}} \lambda x.c_2 \qquad c_i \mapsto_{\mathcal{R}} d$$

Indeed,

$$p \not\lesssim_{\mathfrak{CR}} \lambda x.c_i \qquad \text{but} \qquad \lambda x.c_i \lesssim_{\mathcal{SN}} p$$

Outline

Constructor Rewriting

- **Constructors** are symbols c of type

$$\frac{\Gamma \vdash t_1 : T_1 \quad \dots \quad \Gamma \vdash t_n : T_n}{\Gamma \vdash \mathsf{c}(t_1, \dots, t_n) : \mathsf{B}}$$

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- **Rewrite Rules**

$$\mathsf{f}(p_1, \dots, p_n) \quad \mapsto_{\mathcal{R}} \quad r$$

where $p_1, \dots, p_n$ are constructor patterns.

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$$\lambda x.t \qquad\qquad \mathsf{c}(t_1, \ldots, t_n)$$

External Redexes (CCERSs) [KOO01]

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- A redex (position) u is **External** in a term t
  if no residual of u occurs in a redex argument of a reduct of t.
- If $t \to_{\beta\mathcal{R}} u$ by contracting an external redex of t,
  then u is an **External Reduct** of t.

External Reducts are Strong Principal [Rib08]

Let $\mathcal{R}$ be an **orthogonal** constructor rewrite system.

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Lemma

Let t *be a* ***neutral*** *term and* u *be an* ***external*** *reduct of* t*.*

If t *reduces to a value* v *then* u *reduces to* v*.*

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Let $\mathcal{R}$ be an **orthogonal** constructor rewrite system.

Lemma

Let t be a neutral term and u be an external reduct of t.

If t reduces to a value v then u reduces to v.

- If t is **neutral** and u is an **external** reduct of t then $t \lesssim_{\mathcal{CR}} u$.

External Reducts are Strong Principal [Rib08]

Let $\mathcal{R}$ be an orthogonal constructor rewrite system.

Lemma

Let t be a neutral term and u be an external reduct of t.

If t reduces to a value v then u reduces to v.

- If t is neutral and u is an external reduct of t then $t \lesssim_{\mathcal{CR}} u$.
- **Theorem [KOO01]**
  If $\mathcal{R}$ is **orthogonal**
  then every reducible term has an **external** redex.

External Reducts are Strong Principal [Rib08]

Let $\mathcal{R}$ be an orthogonal constructor rewrite system.

Lemma

Let t be a neutral term and u be an external reduct of t.

If t reduces to a value v then u reduces to v.

- If t is neutral and u is an external reduct of t then $t \lesssim_{\mathfrak{CR}} u$.
- **Theorem [KOO01]**
  If $\mathcal{R}$ is orthogonal
  then every reducible term has an external redex.

Corollary

*If $\mathcal{R}$ is an **orthogonal constructor** rewrite system*
then $\mathfrak{CR}$ is stable by union.

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- For the combination of λ-calculus with orthogonal constructor rewriting, Girard's Candidates are stable by union.
- In [Rib07b], we studied a type system with $(\sqcup \mathrm{E})$ such that for simple rewrite systems $\mathcal{R}$, the following are equivalent:
  (i) terms typable using $(\sqcup \mathrm{E})$ are Strongly Normalizing,
  (ii) the interpretation $(\!|\_|\!) : \mathcal{T} \to \mathcal{P}^\star(\mathcal{SN})^{\perp\!\!\perp\perp\!\!\perp}$ is adequate.

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More generally, can be an interesting point of view to connect notions from denotational semantics, rewriting theory and computational interpretations of logics.

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In particular:

- Logical account of stability by union: can λ-calculus for classical logic admit stable by union type interpretations ?
- Link with notions such as sequentiality and stability.
- Connect typability (eq. with union/intersection types) to properties such as standardization.

Thank you for your attention !

http://www.loria.fr/~riba/

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