The Dual of Pattern Matching - Copattern Matching

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From Codata to Coalgebras

Algebras and Coalgebras

Patterns and Copatterns

Codata types and Decidable Equality

Reduction of Mixed Pattern/Copattern Matching to Operators

Conclusion

Coalgebras in Functional Programming

- Originally functional programming based on
  - function types,
  - inductive data types.
- In computer science, many computations are interactive.
- Since interactions might go on forever (if not terminated by the user), they correspond to non-wellfounded data types
  - Streams, which are infinite lists,
  - non-wellfounded trees (IO-trees).

Codata Type

- Idea of Codata Types:

$$\begin{array}{l} \text{codata Stream} : \text{Set where} \\ \quad \text{cons} \quad : \quad \mathbb{N} \to \text{Stream} \to \text{Stream} \end{array}$$

- Same definition as inductive data type but we are allowed to have infinite chains of constructors

$$\text{cons}\ n_0\ (\text{cons}\ n_1\ (\text{cons}\ n_2\ \cdots))$$

- **Problem 1:** Non-normalisation.
- **Problem 2:** Equality between streams is equality between all elements, and therefore undecidable.
- **Problem 3:** Underlying assumption is

$$\forall s : \text{Stream}.\exists n, s'.s = \text{cons}\ n\ s'$$

which results in undecidable equality.

Subject Reduction Problem

- In order to repair problem of normalisation restrictions on reductions were introduced.
- Resulted in Coq in a long known problem of **subject reduction**.
- In order to avoid this, in Agda dependent elimination for coalgebras disallowed.
  - Makes it difficult to use.

Problem of Subject reduction:

$$
\begin{array}{l}
\text{data } \_{==}\_ \ \{A : \mathrm{Set}\}\ (a : A) : A \to \mathrm{Set} \text{ where} \\
\quad \mathrm{refl} : a == a
\end{array}
$$

$$
\begin{array}{l}
\text{codata Stream} : \mathrm{Set} \text{ where} \\
\quad \mathrm{cons} : \mathbb{N} \to \mathrm{Stream} \to \mathrm{Stream}
\end{array}
$$

$$
\begin{array}{l}
\mathsf{zeros} : \mathrm{Stream} \\
\mathsf{zeros} = \mathrm{cons}\ 0\ \mathsf{zeros}
\end{array}
$$

$$
\begin{array}{l}
\mathsf{force} : \mathrm{Stream} \to \mathrm{Stream} \\
\mathsf{force}\ s = \mathrm{case}\ s\ \mathrm{of}\ (\mathrm{cons}\ x\ y) \to \mathrm{cons}\ x\ y
\end{array}
$$

$$
\begin{array}{l}
\mathsf{lem1} : (s : \mathrm{Stream}) \to s == \mathsf{force}(s)) \\
\mathsf{lem1}\ s = \mathrm{case}\ s\ \mathrm{of}\ (\mathrm{cons}\ x\ y) \to \mathrm{refl}
\end{array}
$$

$$
\begin{array}{l}
\mathsf{lem2} : \mathsf{zeros} == \mathrm{cons}\ 0\ \mathsf{zeros} \\
\mathsf{lem2} = \mathsf{lem1}\ \mathsf{zeros} \\
\mathsf{lem2} \longrightarrow \mathrm{refl} \quad \text{but } \neg(\mathrm{refl} : \mathsf{zeros} == \mathrm{cons}\ 0\ \mathsf{zeros})
\end{array}
$$

Coalgebraic Formulation of Coalgebras

- Solution is to follow the long established categorical formulation of coalgebras.

Initial F-Algebras

- Inductive data types correspond to initial F-Algebras.
- E.g. the natural numbers can be formulated as

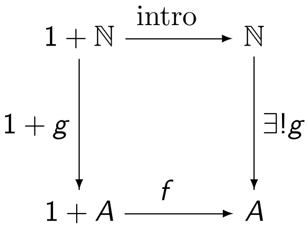
$$
\begin{array}{l}
F(X) = 1 + X \\
\mathrm{intro} : F(\mathbb{N}) \rightarrow \mathbb{N} \\
\mathrm{intro}\ (\mathrm{inl}\ *) \quad = \quad 0 \\
\mathrm{intro}\ (\mathrm{inl}\ n) \quad = \quad \mathrm{S}\ n
\end{array}
$$

and we get the diagram

$$
\begin{array}{ccccc}
1 + \mathbb{N} & = & F(\mathbb{N}) & \xrightarrow{\mathrm{intro}} & \mathbb{N} \\
 & & {\scriptstyle 1 + g = F(g)} \downarrow & & \downarrow {\scriptstyle \exists! g} \\
1 + A & = & F(A) & \xrightarrow{f} & A
\end{array}
$$

Iteration

Existence of unique g corresponds to unique iteration (example $\mathbb{N}$):



$$
\begin{array}{lllll}
g\ 0 & = & g\ (\mathrm{intro\ inl}) & = & f\ \mathrm{inl} \\
g\ (\mathrm{S}\ n) & = & g\ (\mathrm{intro}\ (\mathrm{inr}\ n)) & = & f\ (\mathrm{inr}\ (g\ n))
\end{array}
$$

By choosing arbitrary f we can define g by pattern matching on its argument n:

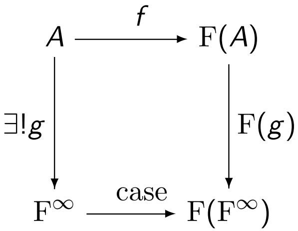
$$
\begin{array}{lll}
g\ 0 & = & a_0 \\
g\ (\mathrm{S}\ n) & = & f\ (g\ n) \text{ for some } f : \mathbb{N} \to \mathbb{N}
\end{array}
$$

Recursion and Induction

- From the principle of unique iteration one can derive the principle of recursion:
  Assume
  $$\begin{array}{lcl} a_0 & : & A \\ f_0 & : & \mathbb{N} \to A \to A \end{array}$$
  We can then define $g : \mathbb{N} \to A$ s.t.
  $$\begin{array}{lcl} g\ 0 & = & a_0 \\ g\ (\mathrm{S}\ n) & = & f_0\ n\ (g\ n) \end{array}$$
- Induction is as recursion but now
  $$g : (n : \mathbb{N}) \to A\ n$$

Coalgebras

Final coalgebras F^∞ are obtained by reversing the arrows in the diagram for F-algebras:



Coalgebras

Consider Streams $= \mathrm{F}^\infty$ where $\mathrm{F}(X) = \mathbb{N} \times X$:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & \mathbb{N} \times A \\ {\scriptstyle \exists ! g}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times g} \\ \mathrm{Stream} & \xrightarrow{\mathrm{case}} & \mathbb{N} \times \mathrm{Stream} \end{array}$$

Let

$$\mathrm{case}\ s \quad = \quad \langle \mathrm{head}\ s, \mathrm{tail}\ s \rangle$$

and

$$f \ a = \langle f_0 \ a, f_1 \ a \rangle$$

Guarded Recursion

$$
\begin{array}{ccc}
A & \xrightarrow{\langle f_0, f_1 \rangle} & \mathbb{N} \times A \\
\downarrow{\scriptstyle \exists! g} & & \downarrow{\scriptstyle \mathrm{id} \times g} \\
\mathrm{Stream} & \xrightarrow{\langle \mathrm{head}, \mathrm{tail} \rangle} & \mathbb{N} \times \mathrm{Stream}
\end{array}
$$

Resulting equations:

$$
\begin{array}{lcl}
\mathrm{head}\ (g\ a) & = & f_0\ a \\
\mathrm{tail}\ \ (g\ a) & = & g\ (f_1\ a)
\end{array}
$$

Example of Guarded Recursion

$$\begin{array}{lcl} \mathrm{head}\ (g\ a) & = & f_0\ a \\ \mathrm{tail}\ \ (g\ a) & = & g\ (f_1\ a) \end{array}$$

describes a schema of guarded recursion (or better coiteration)
As an example, with $A = \mathbb{N}$, $f_0\ n = n$, $f_1\ n = n + 1$ we obtain:

$$\begin{array}{lcl} \multicolumn{3}{l}{\mathrm{inc} : \mathbb{N} \to \mathrm{Stream}} \\ \mathrm{head}\ (\mathrm{inc}\ n) & = & n \\ \mathrm{tail}\ \ (\mathrm{inc}\ n) & = & \mathrm{inc}\ (n + 1) \end{array}$$

Corecursion

In coiteration we need to make in tail always a recursive call:

$$\mathrm{tail}\ (g\ a) = g\ (f_1\ a)$$

Corecursion allows for tail to escape into a previously defined stream.
Assume

$$\begin{array}{lcl} A & : & \mathrm{Set} \\ f_0 & : & A \to \mathbb{N} \\ f_1 & : & A \to (\mathrm{Stream} + A) \end{array}$$

we get $g : A \to \mathrm{Stream}$ s.t.

$$\begin{array}{lllcll} \mathrm{head} & (g\ a) & = & f_0\ a & & \\ \mathrm{tail} & (g\ a) & = & s & \text{if} & f_1\ a = \mathrm{inl}\ s \\ \mathrm{tail} & (g\ a) & = & g\ a' & \text{if} & f_1\ a = \mathrm{inr}\ a' \end{array}$$

Definition of cons by Corecursion

$$
\begin{array}{lllll}
\mathrm{head}\ (g\ a) & = & f_0\ a & & \\
\mathrm{tail}\ \ (g\ a) & = & s & \text{if} & f_1\ a = \mathrm{inl}\ s \\
\mathrm{tail}\ \ (g\ a) & = & g\ a' & \text{if} & f_1\ a = \mathrm{inr}\ a'
\end{array}
$$

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathrm{cons} : \mathbb{N} \to \mathrm{Stream} \to \mathrm{Stream}} \\
\mathrm{head}\ (\mathrm{cons}\ n\ s) & = & n \\
\mathrm{tail}\ \ (\mathrm{cons}\ n\ s) & = & s
\end{array}
$$

Nested Corecursion

$$
\begin{array}{lcl}
\multicolumn{3}{l}{\mathrm{stutter} : \mathbb{N} \to \mathrm{Stream}} \\
\mathrm{head}\ \ \ \ \ \ \ (\mathrm{stutter}\ n) & = & n \\
\mathrm{head}\ (\mathrm{tail}\ (\mathrm{stutter}\ n)) & = & n \\
\mathrm{tail}\ \ \ (\mathrm{tail}\ (\mathrm{stutter}\ n)) & = & \mathrm{stutter}\ (n+1)
\end{array}
$$

Even more general schemata can be defined.

Definition of Coalgebras by Observations

- We see now that elements of coalgebras are defined by their observations:
  An element s of Stream is given by defining

$$\begin{array}{lll} \mathrm{head} & s & : & \mathbb{N} \\ \mathrm{tail} & s & : & \mathrm{Stream} \end{array}$$

- This generalises the function type.
  Functions $f : A \to B$ are as well determined by observations, namely by defining

$$f \ a : B$$

  - An $f : A \to B$ is any program which applied to $a : A$ returns some $b : B$.
- **Inductive data types** are defined by **construction**
  coalgebraic data types and **functions** by **observations**.

Relationship to Objects in Object-Oriented Programming

- Objects in Object-Oriented Programming are types which are defined by their observations.
- Therefore objects are coalgebraic types by nature.

Weakly Final Coalgebra

- Equality for final coalgebras is undecidable:
  Two streams

$$\begin{array}{ccccccccc} s & = & (a_0 & , & a_1 & , & a_2 & , & \ldots \\ t & = & (b_0 & , & b_1 & , & b_2 & , & \ldots \end{array}$$

  are equal iff $a_i = b_i$ for all i.
- Even the weak assumption

$$\forall s.\exists n, s'.s = \mathrm{cons}\ n\ s'$$

  results in an undecidable equality.
- Weakly final coalgebras obtained by omitting uniqueness of g in diagram for coalgebras.
- However, one can extend schema of coiteration as above, and still preserve decidability of equality.
  - Those schemata are usually not derivable in weakly final coalgebras.

Patterns and Copatterns

- We can define now functions by patterns and copatterns.
- Example define stream:
  $f\ \ n =$
  $n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{lcl} f & : & \mathbb{N} \to \mathrm{Stream} \\ f & = & ? \end{array}$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{lcl} f & : & \mathbb{N} \to \mathrm{Stream} \\ f & = & ? \end{array}$$

Copattern matching on $f : \mathbb{N} \to \mathrm{Stream}$:

$$\begin{array}{lcl} f & : & \mathbb{N} \to \mathrm{Stream} \\ f\ n & = & ? \end{array}$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{lcl} f : \mathbb{N} \to \mathrm{Stream} \\ f\ n & = & ? \end{array}$$

Copattern matching on $f\ n : \mathrm{Stream}$:

$$\begin{array}{lcl} f : \mathbb{N} \to \mathrm{Stream} \\ \mathrm{head}\ (f\ n) & = & ? \\ \mathrm{tail}\ \ (f\ n) & = & ? \end{array}$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l}f : \mathbb{N} \to \mathrm{Stream} \\ f\ n \quad = \quad ?\end{array}$$

Solve first case, copattern match on second case:

$$\begin{array}{lcl}f : \mathbb{N} \to \mathrm{Stream} \\ \mathrm{head}\ \qquad (f\ n) & = & n \\ \mathrm{head}\ (\mathrm{tail}\ (f\ n)) & = & ? \\ \mathrm{tail}\ \ (\mathrm{tail}\ (f\ n)) & = & ?\end{array}$$

Patterns and Copatterns

$f\ n = n, n, n{-}1, n{-}1, \ldots 0, 0, N, N, N{-}1, N{-}1, \ldots 0, 0, N, N, N{-}1, N{-}1,$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f\ n \quad = \quad ? \end{array}$$

Solve second line, pattern match on n

$$\begin{array}{lcl} f : \mathbb{N} \to \mathrm{Stream} & & \\ \mathrm{head}\ \ (f\ n) & = & n \\ \mathrm{head}\ (\mathrm{tail}\ (f\ n)) & = & n \\ \mathrm{tail}\ \ (\mathrm{tail}\ (f\ 0)) & = & ? \\ \mathrm{tail}\ \ (\mathrm{tail}\ (f\ (\mathrm{S}\ n))) & = & ? \end{array}$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f\ n \quad = \quad ? \end{array}$$

Solve remaining cases

$$\begin{array}{llcl} f : \mathbb{N} \to \mathrm{Stream} & & & \\ \mathrm{head} & (f\ n) & = & n \\ \mathrm{head} & (\mathrm{tail}\ (f\ n)) & = & n \\ \mathrm{tail} & (\mathrm{tail}\ (f\ 0)) & = & f\ N \\ \mathrm{tail} & (\mathrm{tail}\ (f\ (\mathrm{S}\ n))) & = & f\ n \end{array}$$

Results of paper in POPL (2013)

- Development of a recursive simply typed calculus (no termination check).
- Allows to derive schemata for pattern/copattern matching.
- Proof that subject reduction holds.

$$t : A, \quad t \longrightarrow t' \text{ implies } t' : A$$

Theorem Regarding Undecidabilty of Equality

Theorem

Assume the following:

- *There exists a subset* $\mathrm{Stream} \subseteq \mathbb{N}$,
- *computable functions*
  $\mathrm{head} : \mathrm{Stream} \to \mathbb{N}$, $\mathrm{tail} : \mathrm{Stream} \to \mathrm{Stream}$,
- *a decidable equality* $\_ == \_$ *on* Stream *which is congruence,*
- *the possibilty to define elements of* Stream *by guarded recursion based on primitive recursive functions* $f, g : \mathbb{N} \to \mathbb{N}$, *such that the standard equalities related to guarded recursion hold.*

Then it is not possible to fulfil the following condition:

$$\forall s, s' : \mathrm{Stream}.\mathrm{head}\ s = \mathrm{head}\ s' \wedge \mathrm{tail}\ s == \mathrm{tail}\ s' \to s == s' \qquad (*)$$

Consequences for Codata Approach

Remark

Condition $(*)$ *is fulfilled if we have an operation*
$\mathrm{cons} : \mathbb{N} \to \mathrm{Stream} \to \mathrm{Stream}$ *preserving equalities s.t.*

$$\forall s : \mathrm{Stream}. s = \mathrm{cons}\ (\mathrm{head}\ s)\ (\mathrm{tail}\ s)$$

So we cannot have a type theory with streams, decidable type checking and decidable equality on streams such that

$$\forall s. \exists n, s'. s == \mathrm{cons}\ n\ s'$$

as assumed by the codata approach.

Proof of Theorem

- Assume we had the above.
- By

$$s \approx n_0 :: n_1 :: n_2 :: \cdots n_k :: s'$$

  we mean the equations using head, tail expressing that s behaves as the stream indicated on the right hand side.
- Define by guarded recursion $l : \mathrm{Stream}$

$$l \approx 1 :: 1 :: 1 :: \cdots$$

Proof of Theorem

- For e code for a Turing machine define by guarded recursion based on primitive recursion functions f, g s.t. if e terminates after n steps and returns result k then

$$f\ e \approx \underbrace{0 :: 0 :: 0 :: \cdots :: 0}_{n \text{ times}} :: l$$

$$g\ e \approx \begin{cases} \underbrace{0 :: 0 :: 0 :: \cdots :: 0}_{n \text{ times}} :: l & \text{if } k = 0 \\ \underbrace{0 :: 0 :: 0 :: \cdots :: 0}_{n+1 \text{ times}} :: l & \text{if } k > 0 \end{cases}$$

Proof of Theorem

$$f\ e \quad \approx \quad \underbrace{0 :: 0 :: 0 :: \cdots :: 0}_{n \text{ times}} :: l$$

$$g\ e \quad \approx \quad \begin{cases} \underbrace{0 :: 0 :: 0 :: \cdots :: 0}_{n \text{ times}} :: l & \text{if } k = 0 \\ \underbrace{0 :: 0 :: 0 :: \cdots :: 0}_{n+1 \text{ times}} :: l & \text{if } k > 0 \end{cases}$$

- If e terminates after n steps with result 0 then

$$f\ e == g\ e$$

- If e terminates after n steps with result > 0 then

$$\neg(f\ e == g\ e)$$

Proof of Theorem

- So

$$\lambda e.(f\ e == g\ e)$$

  separates the TM with result 0 from those with result > 0.
- But these two sets are inseparable.

Operators for Primitive (Co)Recursion

$$
\begin{array}{lcl}
\mathrm{P}_{\mathbb{N},A} : A \to (\mathbb{N} \to A \to A) \to \mathbb{N} \to A \\
\mathrm{P}_{\mathbb{N},A}\ \mathrm{step}_0\ \mathrm{step}_\mathrm{S}\ 0 & = & \mathrm{step}_0 \\
\mathrm{P}_{\mathbb{N},A}\ \mathrm{step}_0\ \mathrm{step}_\mathrm{S}\ (\mathrm{S}\ n) & = & \mathrm{step}_\mathrm{S}\ n\ (\mathrm{P}_{\mathbb{N},A}\ \mathrm{step}_0\ \mathrm{step}_\mathrm{S}\ n)
\end{array}
$$

$$
\begin{array}{lcl}
\mathrm{coP}_{\mathrm{Stream},A} : (A \to \mathbb{N}) \to (A \to (\mathrm{Stream} + A)) \to A \to \mathrm{Stream} \\
\mathrm{head}\ (\mathrm{coP}_{\mathrm{Stream},A}\ \mathrm{step}_\mathrm{head}\ \mathrm{step}_\mathrm{tail}\ a) & = & \mathrm{step}_\mathrm{head}\ a \\
\mathrm{tail}\ \ \ (\mathrm{coP}_{\mathrm{Stream},A}\ \mathrm{step}_\mathrm{head}\ \mathrm{step}_\mathrm{tail}\ a) & = & \\
\quad \mathrm{case}_{\mathrm{Stream},A,\mathrm{Stream}}\ \mathrm{id}\ (\mathrm{coP}_{\mathrm{Stream},A}\ \mathrm{step}_\mathrm{head}\ \mathrm{step}_\mathrm{tail})\ (\mathrm{step}_\mathrm{tail}\ a)
\end{array}
$$

Operators for full/primitive (co)recursion

$$\mathrm{R}_{\mathbb{N},A} : ((\mathbb{N} \to A) \to A) \to ((\mathbb{N} \to A) \to \mathbb{N} \to A) \to \mathbb{N} \to A$$

$$\begin{array}{lcl} \mathrm{R}_{\mathbb{N},A}\ \mathrm{step}_0\ \mathrm{step}_\mathrm{S}\ 0 & = & \mathrm{step}_0\ (\mathrm{R}_{\mathbb{N},A}\ \mathrm{step}_0\ \mathrm{step}_\mathrm{S}) \\ \mathrm{R}_{\mathbb{N},A}\ \mathrm{step}_0\ \mathrm{step}_\mathrm{S}\ (\mathrm{S}\ n) & = & \mathrm{step}_\mathrm{S}\ (\mathrm{R}_{\mathbb{N},A}\ \mathrm{step}_0\ \mathrm{step}_\mathrm{S})\ n \end{array}$$

$$\begin{array}{ll} \mathrm{coR}_{\mathrm{Stream},A} : & ((A \to \mathrm{Stream}) \to A \to \mathbb{N}) \\ & \to ((A \to \mathrm{Stream}) \to A \to \mathrm{Stream}) \\ & \to \mathrm{Stream} \end{array}$$

$$\begin{array}{lcl} \mathrm{head}\ (\mathrm{coR}_{\mathrm{Stream},A}\ \mathrm{step}_\mathrm{head}\ \mathrm{step}_\mathrm{tail}\ a) & = & \mathrm{step}_\mathrm{head} \\ & & (\mathrm{coR}_{\mathrm{Stream},A}\ \mathrm{step}_\mathrm{head}\ \mathrm{step}_\mathrm{tail})\ a \\ \mathrm{tail}\ (\mathrm{coR}_{\mathrm{Stream},A}\ \mathrm{step}_\mathrm{head}\ \mathrm{step}_\mathrm{tail}\ a) & = & \mathrm{step}_\mathrm{tail} \\ & & (\mathrm{coR}_{\mathrm{Stream},A}\ \mathrm{step}_\mathrm{head}\ \mathrm{step}_\mathrm{tail})\ a \end{array}$$

Consider Example from above

$$
\begin{array}{llll}
f : \mathbb{N} \to \mathrm{Stream} & & & \\
\mathrm{head} & (f\ n) & = & n \\
\mathrm{head}\ (\mathrm{tail}\ (f\ n)) & & = & n \\
\mathrm{tail} & (\mathrm{tail}\ (f\ 0)) & = & f\ N \\
\mathrm{tail} & (\mathrm{tail}\ (f\ (\mathrm{S}\ n))) & = & f\ n
\end{array}
$$

This example can be reduced to primitive (co)recursion.
Step 1: Following the development of the (co)pattern matching definition, unfold it into simulteneous non-nested (co)pattern matching definitions.

Step 1: Unnesting of Nested (Co)Pattern Matching

We follow the steps in the pattern matching:
We start with

$$
\begin{array}{l}
f : \mathbb{N} \to \text{Stream} \\
\text{head}\ (f\ \ n) \quad = \quad n \\
\text{tail}\ \ \ (f\ \ n) \quad = \quad ?
\end{array}
$$

Copattern matching on $\mathrm{tail}\ (f\ n)$:

$$\begin{array}{llll} f : \mathbb{N} \to \mathrm{Stream} & & & \\ \mathrm{head} & (f\ n) & = & n \\ \mathrm{head}\ (\mathrm{tail} & (f\ n) & = & n \\ \mathrm{tail}\ \ (\mathrm{tail} & (f\ n) & = & ? \end{array}$$

corresponds to

$$\begin{array}{llll} f : \mathbb{N} \to \mathrm{Stream} & & & \\ \mathrm{head} & (f\ n) & = & n \\ \mathrm{tail} & (f\ n) & = & g\ n \end{array}$$

$$\begin{array}{lllll} & g : \mathbb{N} \to \mathrm{Stream} & & & \\ (\mathrm{head}\ (\mathrm{tail}\ (f\ n)) \quad =) & \mathrm{head} & (g\ n) & = & n \\ (\mathrm{tail}\ \ (\mathrm{tail}\ (f\ n)) \quad =) & \mathrm{tail} & (g\ n) & = & ? \end{array}$$

Pattern matching on tail (tail (f n)):

$$
\begin{array}{lll}
f : \mathbb{N} \to \mathrm{Stream} & & \\
\mathrm{head}\ \ (f\ n) & = & n \\
\mathrm{head}\ (\mathrm{tail}\ (f\ n) & = & n \\
\mathrm{tail}\ \ (\mathrm{tail}\ (f\ 0) & = & f\ N \\
\mathrm{tail}\ \ (\mathrm{tail}\ (f\ (\mathrm{S}\ n)) & = & f\ n
\end{array}
$$

corresponds to

$$
\begin{array}{llll}
 & f : \mathbb{N} \to \mathrm{Stream} & & \\
 & \mathrm{head}\ (f\ n) & = & n \\
 & \mathrm{tail}\ \ (f\ n) & = & g\ n \\
 & & & \\
 & g : \mathbb{N} \to \mathrm{Stream} & & \\
(\mathrm{head}\ (\mathrm{tail}\ (f\ n)) \quad =) & \mathrm{head}\ (g\ n) & = & n \\
(\mathrm{tail}\ \ (\mathrm{tail}\ (f\ n)) \quad =) & \mathrm{tail}\ \ (g\ n) & = & k\ n \\
 & & & \\
 & k : \mathbb{N} \to \mathrm{Stream} & & \\
(\mathrm{tail}\ \ (\mathrm{tail}\ (f\ 0)) \quad =) & k\ \ 0 & = & f\ N \\
(\mathrm{tail}\ \ (\mathrm{tail}\ (f\ (\mathrm{S}\ n))) \quad =) & k\ \ (\mathrm{S}\ n) & = & f\ n
\end{array}
$$

Step 2: Reduction to Primitive (Co)recursion

- This can now easily be reduced to full (co)recursion.
- In this example we can reduce it to primitive (co)recursion.
- First combine f, g into one function $f + g$.

$$
\begin{aligned}
&f : \mathbb{N} \to \mathrm{Stream} \\
&f\ n && = && (f+g)\ (\underline{\mathrm{f}}\ n) \\
\\
&(f+g) : (\underline{\mathrm{f}}(\mathbb{N}) + \underline{\mathrm{g}}(\mathbb{N})) \to \mathrm{Stream} \\
&\mathrm{head} \quad ((f+g)\ (\underline{\mathrm{f}}\ n)) && = && n \\
&\mathrm{head} \quad ((f+g)\ (\underline{\mathrm{g}}\ n)) && = && n \\
&\mathrm{tail} \quad ((f+g)\ (\underline{\mathrm{f}}\ n)) && = && (f+g)\ (\underline{\mathrm{g}}\ n) \\
&\mathrm{tail} \quad ((f+g)\ (\underline{\mathrm{f}}\ n)) && = && k\ n \\
\\
&k : \mathbb{N} \to \mathrm{Stream} \\
&k\ 0 && = && (f+g)\ (\underline{\mathrm{f}}\ N) \\
&k\ (\mathrm{S}\ n) && = && (f+g)\ (\underline{\mathrm{f}}\ n)
\end{aligned}
$$

Unfolding of the Pattern Matchings

- The call of k has result always of the form $(f + g)(\ fbf\ \ n))$.
  So we can replace the recursive call $k\ n$ by $(f + g)(\underline{\mathrm{f}}\ (k'\ n))$.

$$
\begin{aligned}
&f : \mathbb{N} \to \mathrm{Stream} \\
&f\ n && = && (f+g)\ (\underline{\mathrm{f}}\ n) \\
\\
&(f+g) : (\underline{\mathrm{f}}(\mathbb{N}) + \underline{\mathrm{g}}(\mathbb{N})) \to \mathrm{Stream} \\
&\mathrm{head}\quad ((f+g)\ (\underline{\mathrm{f}}\ n)) && = && n \\
&\mathrm{head}\quad ((f+g)\ (\underline{\mathrm{g}}\ n)) && = && n \\
&\mathrm{tail}\quad ((f+g)\ (\underline{\mathrm{f}}\ n)) && = && (f+g)\ (\underline{\mathrm{g}}\ n) \\
&\mathrm{tail}\quad ((f+g)\ (\underline{\mathrm{f}}\ n)) && = && (f+g)\ (\underline{\mathrm{f}}\ (k'\ n)) \\
\\
&k' : \mathbb{N} \to \mathbb{N} \\
&k\ 0 && = && N \\
&k\ (\mathrm{S}\ n) && = && n
\end{aligned}
$$

Unfolding of the Pattern Matchings

- $(f + g)$ can be defined by primitive corecursion.
- k' can be defined by primitive recursion.

$$
\begin{aligned}
&f : \mathbb{N} \to \mathrm{Stream}\\
&f\ n \quad = \quad (f+g)\ (\underline{\mathrm{f}}\ n)
\end{aligned}
$$

$$
\begin{aligned}
&(f+g) : (\underline{\mathrm{f}}(\mathbb{N}) + \underline{\mathrm{g}}(\mathbb{N})) \to \mathrm{Stream}\\
&(f+g) =\\
&\quad \mathrm{coP}_{\mathrm{Stream},(\underline{\mathrm{f}}(\mathbb{N})+\underline{\mathrm{g}}(\mathbb{N}))}\ (\lambda x.\mathrm{case}_r(x)\ \mathrm{of}\\
&\qquad\qquad (\underline{\mathrm{f}}\ n) \quad \longrightarrow \quad n\\
&\qquad\qquad (\underline{\mathrm{g}}\ n) \quad \longrightarrow \quad n)\\
&\quad\ (\lambda x.\mathrm{case}_r(x)\ \mathrm{of}\\
&\qquad\qquad (\underline{\mathrm{f}}\ n) \quad \longrightarrow \quad \underline{\mathrm{g}}\ n\\
&\qquad\qquad (\underline{\mathrm{g}}\ n) \quad \longrightarrow \quad \underline{\mathrm{f}}\ (k'\ n))
\end{aligned}
$$

$$
\begin{aligned}
&k' : \mathbb{N} \to \mathbb{N}\\
&k' = \mathrm{P}_{\mathbb{N},\mathbb{N}}\ N\ (\lambda n, ih.n)
\end{aligned}
$$

Reduction to Primitive (Co)Recursion

- The case distinction can be trivially replaced by the case distinction operator.

$$f : \mathbb{N} \to \mathrm{Stream}$$
$$f\ n \quad = \quad (f + g)\ (\underline{\mathrm{f}}\ n)$$

$$(f + g) : (\underline{\mathrm{f}}(\mathbb{N}) + \underline{\mathrm{g}}(\mathbb{N})) \to \mathrm{Stream}$$
$$\begin{aligned}(f + g) =&\\ &\mathrm{coP}_{\mathrm{Stream}, \underline{\mathrm{f}}(\mathbb{N}) + \underline{\mathrm{g}}(\mathbb{N})}\ (\mathrm{case}_{\underline{\mathrm{f}}(\mathbb{N}) + \underline{\mathrm{g}}(\mathbb{N})}\ \mathrm{id}\ \mathrm{id})\\ &\qquad (\mathrm{case}_{\underline{\mathrm{f}}(\mathbb{N}) + \underline{\mathrm{g}}(\mathbb{N})}\ \underline{\mathrm{g}}\ (\underline{\mathrm{f}} \circ k'))\end{aligned}$$

$$k' : \mathbb{N} \to \mathbb{N}$$
$$k' = \mathrm{P}_{\mathbb{N}, \mathbb{N}}\ N\ (\lambda n, ih.n)$$

Conclusion

- Codata types make the assumption

$$\forall s : \mathrm{Stream}.\exists n, s'.s = \mathrm{cons}\ n\ s'$$

  which cannot be combined with a decidable equality.
- In general Codata types cause problems such as subject reduction.
- Solution:
  - Coalgebra are determined by their elimination rule.
  - Introduction rule corresponds to copattern matching.
- Solves problem of subject reduction.

Conclusion

- One can reduce certain cases of recursive nested (co)pattern matching to primitive (co)recursion.
  - Systematic treatment needs still to be done.
  - Cases which can be reduced should be those to be accepted by a termination checker.
  - If the reduction succeeds we get a normalising version (by Mendler and Geuvers).
  - Therefore a termination checked version of the calculus is normalising.