The Dual of Pattern Matching - Copattern Matching

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From Codata to Coalgebras

Algebras and Coalgebras

Patterns and Copatterns

Codata types and Decidable Equality

Reduction of Mixed Pattern/Copattern Matching to Operators

Conclusion
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Coalgebras in Functional Programming

- Originally functional programming based on
  - function types,
  - inductive data types.

- In computer science, many computations are interactive.

- Since interactions might go on forever (if not terminated by the user), they correspond to non-wellfounded data types
  - Streams, which are infinite lists,
  - non-wellfounded trees (IO-trees).
Codata Type

- Idea of Codata Types:

\[
\text{codata Stream : Set where} \\
\text{cons : } \mathbb{N} \to \text{Stream} \to \text{Stream}
\]

- Same definition as inductive data type but we are allowed to have infinite chains of constructors

\[
\text{cons } n_0 \ (\text{cons } n_1 \ (\text{cons } n_2 \ \cdots))
\]

- **Problem 1:** Non-normalisation.

- **Problem 2:** Equality between streams is equality between all elements, and therefore undecidable.

- **Problem 3:** Underlying assumption is

\[
\forall s : \text{Stream. } \exists n, s'. s = \text{cons } n \ s'
\]

which results in undecidable equality.
In order to repair problem of normalisation restrictions on reductions were introduced.

Resulted in Coq in a long known problem of subject reduction.

In order to avoid this, in Agda dependent elimination for coalgebras disallowed.
  - Makes it difficult to use.
Problem of Subject reduction:

\[
\text{data } _==_ \{ A : \text{Set} \} (a : A) : A \rightarrow \text{Set where} \\
\text{refl} : a == a
\]

\text{codata Stream : Set where} \\
\text{cons} : \mathbb{N} \rightarrow \text{Stream} \rightarrow \text{Stream}

\text{zeros : Stream} \\
\text{zeros = cons 0 zeros}

\text{force : Stream \rightarrow Stream} \\
\text{force } s = \text{case } s \text{ of } (\text{cons } x \ y) \rightarrow \text{cons } x \ y

\text{lem1 : } (s : \text{Stream}) \rightarrow s == \text{force}(s)) \\
\text{lem1 } s = \text{case } s \text{ of } (\text{cons } x \ y) \rightarrow \text{refl}

\text{lem2 : zeros == cons 0 zeros} \\
\text{lem2 = lem1 zeros} \\
\text{lem2 \rightarrow refl \ but \ \neg(\text{refl} : \text{zeros == cons 0 zeros})}
Solution is to follow the long established categorical formulation of coalgebras.
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Inductive data types correspond to initial F-Algebras.

E.g. the natural numbers can be formulated as

\[
F(X) = 1 + X
\]

\[
\text{intro} : F(\mathbb{N}) \to \mathbb{N}
\]

\[
\text{intro (inl } \ast \text{ ) } = 0
\]

\[
\text{intro (inl } n \text{ ) } = S \ n
\]

and we get the diagram

\[
\begin{array}{ccc}
1 + \mathbb{N} & = & 1 + g = F(g) \\
 & & 1 + A = 1 + A \\
 & & \exists! g
\end{array}
\]

\[
\begin{array}{ccc}
& & f \\
\downarrow & & \downarrow \exists! g \\
F(\mathbb{N}) & \xrightarrow{\text{intro}} & \mathbb{N} \\
& & \downarrow \\
& & A
\end{array}
\]
Iteration

Existence of unique $g$ corresponds to unique iteration (example $\mathbb{N}$):

$$
1 + \mathbb{N} \xrightarrow{\text{intro}} \mathbb{N}
$$

$$
1 + g \quad \exists! g
$$

$$
1 + \mathbb{A} \xrightarrow{f} \mathbb{A}
$$

$$
g \ 0 \quad = \quad g \ (\text{intro} \ \text{inl}) \quad = \quad f \ \text{inl}
$$

$$
g \ (S \ n) \quad = \quad g \ (\text{intro} \ (\text{inr} \ n)) \quad = \quad f \ (\text{inr} \ (g \ n))
$$

By choosing arbitrary $f$ we can define $g$ by pattern matching on its argument $n$:

$$
g \ 0 \quad = \quad a_0
$$

$$
g \ (S \ n) \quad = \quad f \ (g \ n) \text{ for some } f : \mathbb{N} \to \mathbb{N}$$
Recursion and Induction

- From the principle of unique iteration one can derive the principle of recursion:
  Assume
  
  \[
  a_0 : A \\
  f_0 : \mathbb{N} \rightarrow A \rightarrow A
  \]

  We can then define \( g : \mathbb{N} \rightarrow A \) s.t.
  
  \[
  g \ 0 \ = \ a_0 \\
  g \ (S \ n) \ = \ f_0 \ n \ (g \ n)
  \]

- Induction is as recursion but now
  
  \[
  g : (n : \mathbb{N}) \rightarrow A \ n
  \]
Coalgebras

Final coalgebras $F^\infty$ are obtained by reversing the arrows in the diagram for $F$-algebras:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & F(A) \\
\downarrow{\exists!g} & & \downarrow{F(g)} \\
F^\infty & \xrightarrow{\text{case}} & F(F^\infty)
\end{array}
$$
Consider Streams = \( F^{\infty} \) where \( F(X) = \mathbb{N} \times X \):

\[
\begin{align*}
A & \xrightarrow{f} \mathbb{N} \times A \\
\exists! g & \quad \text{case} \quad \text{id} \times g \\
\text{Stream} & \xrightarrow{\text{case}} \mathbb{N} \times \text{Stream}
\end{align*}
\]

Let

\[
\text{case } s = \langle \text{head } s, \text{tail } s \rangle
\]

and

\[
f a = \langle f_0 a, f_1 a \rangle
\]
Guarded Recursion

\[ A \xrightarrow{\langle f_0, f_1 \rangle} \mathbb{N} \times A \]

\[ \exists! g \]

\[ \text{Stream} \xrightarrow{\langle \text{head}, \text{tail} \rangle} \mathbb{N} \times \text{Stream} \]

Resulting equations:

\[
\begin{align*}
\text{head} \ (g \ a) &= f_0 \ a \\
\text{tail} \ (g \ a) &= g \ (f_1 \ a)
\end{align*}
\]
Example of Guarded Recursion

\[
\begin{align*}
\text{head} (g \ a) &= f_0 \ a \\
\text{tail} (g \ a) &= g (f_1 \ a)
\end{align*}
\]

describes a schema of guarded recursion (or better coiteration).

As an example, with \( A = \mathbb{N} \), \( f_0 \ n = n \), \( f_1 \ n = n + 1 \) we obtain:

\[
\begin{align*}
\text{inc} : \mathbb{N} &\to \text{Stream} \\
\text{head} (\text{inc} \ n) &= n \\
\text{tail} (\text{inc} \ n) &= \text{inc} (n + 1)
\end{align*}
\]
Corecursion

In coiteration we need to make in tail always a recursive call:

\[ \text{tail } (g \ a) = g \ (f_1 \ a) \]

Corecursion allows for tail to escape into a previously defined stream. Assume

\[
\begin{align*}
A & : \text{Set} \\
f_0 & : A \to \mathbb{N} \\
f_1 & : A \to (\text{Stream} + A)
\end{align*}
\]

we get \( g : A \to \text{Stream} \) s.t.

\[
\begin{align*}
\text{head } (g \ a) & = f_0 \ a \\
\text{tail } (g \ a) & = s \quad \text{if } f_1 \ a = \text{inl } s \\
\text{tail } (g \ a) & = g \ a' \quad \text{if } f_1 \ a = \text{inr } a'
\end{align*}
\]
Definition of cons by Corecursion

\[
\begin{align*}
\text{head} \ (g \ a) & = \ f_0 \ a \\
\text{tail} \ (g \ a) & = \ s \quad \text{if} \quad f_1 \ a = \text{inl} \ s \\
\text{tail} \ (g \ a) & = \ g \ a' \quad \text{if} \quad f_1 \ a = \text{inr} \ a'
\end{align*}
\]

\[
\begin{align*}
\text{cons} : \mathbb{N} \to \text{Stream} \to \text{Stream} \\
\text{head} \ (\text{cons} \ n \ s) & = \ n \\
\text{tail} \ (\text{cons} \ n \ s) & = \ s
\end{align*}
\]
Nested Corecursion

\[
\text{stutter} : \mathbb{N} \to \text{Stream} \\
\text{head} \ (\text{stutter} \ n) = n \\
\text{head} \ (\text{tail} \ (\text{stutter} \ n)) = n \\
\text{tail} \ (\text{tail} \ (\text{stutter} \ n)) = \text{stutter} \ (n + 1)
\]

Even more general schemata can be defined.
Definition of Coalgebras by Observations

- We see now that elements of coalgebras are defined by their observations:
  An element \( s \) of \( \text{Stream} \) is given by defining
  
  \[
  \begin{align*}
  \text{head} \ s & : \ \mathbb{N} \\
  \text{tail} \ s & : \ \text{Stream}
  \end{align*}
  \]

- This generalises the function type.
  Functions \( f : A \rightarrow B \) are as well determined by observations, namely by defining
  
  \[ f \ a : B \]

- An \( f : A \rightarrow B \) is any program which applied to \( a : A \) returns some \( b : B \).

- **Inductive data types** are defined by construction **coalgebraic data types** and **functions** by observations.
Objects in Object-Oriented Programming are types which are defined by their observations.

Therefore objects are coalgebraic types by nature.
Weakly Final Coalgebra

- Equality for final coalgebras is undecidable:
  Two streams
  \[ s = (a_0, a_1, a_2, \ldots) \]
  \[ t = (b_0, b_1, b_2, \ldots) \]
  are equal iff \( a_i = b_i \) for all \( i \).

- Even the weak assumption
  \[ \forall s. \exists n, s'. s = \text{cons } n \ s' \]
  results in an undecidable equality.

- Weakly final coalgebras obtained by omitting uniqueness of \( g \) in diagram for coalgebras.

- However, one can extend schema of coiteration as above, and still preserve decidability of equality.
  - Those schemata are usually not derivable in weakly final coalgebras.
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We can define now functions by patterns and copatterns.

Example define stream:

\[ f \ n = \]
\[ n, n, n - 1, n - 1, \ldots 0, 0, N, N, N - 1, N - 1, \ldots 0, 0, N, N, N - 1, N - 1, \ldots \]
Patterns and Copatterns

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f = ? \]
Patterns and Copatterns

\[ f \, n = n, \, n, \, n-1, \, n-1, \ldots 0, \, 0, \, N, \, N, \, N-1, \, N-1, \ldots 0, \, 0, \, N, \, N, \, N-1, \, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f \, n \, = \, ? \]

Copattern matching on \( f : \mathbb{N} \rightarrow \text{Stream} \):

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f \, n \, = \, ? \]
$f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

\[ f : \mathbb{N} \rightarrow \text{Stream} \]
\[ f \ n = ? \]

**Copattern matching** on $f \ n : \text{Stream}$:

\[ f : \mathbb{N} \rightarrow \text{Stream} \]
\[ \text{head} \ (f \ n) = ? \]
\[ \text{tail} \ (f \ n) = ? \]
Patterns and Copatterns

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f \ n = ? \]

**Solve first case, copattern match on second case:**

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ \text{head} \ (f \ n) = n \]

\[ \text{head} \ (\text{tail} \ (f \ n)) = ? \]

\[ \text{tail} \ (\text{tail} \ (f \ n)) = ? \]
\( f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \)

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f \ n = ? \]

**Solve second line, pattern match on** \( n \)

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[
\begin{align*}
\text{head} & \ (f \ n) \quad = \quad n \\
\text{head (tail} \ (f \ n)) & \quad = \quad n \\
\text{tail} \ (\text{tail} \ (f \ 0)) & \quad = \quad ? \\
\text{tail} \ (\text{tail} \ (f \ (S \ n))) & \quad = \quad ?
\end{align*}
\]
\( f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \)

\[ f : \mathbb{N} \to \text{Stream} \]
\[ f \ n = ? \]

**Solve remaining cases**

\[ f : \mathbb{N} \to \text{Stream} \]
\[ \text{head} \ (f \ n) = n \]
\[ \text{head} \ (\text{tail} \ (f \ n)) = n \]
\[ \text{tail} \ (\text{tail} \ (f \ 0)) = f \ N \]
\[ \text{tail} \ (\text{tail} \ (f \ (S \ n))) = f \ n \]
Results of paper in POPL (2013)

- Development of a recursive simply typed calculus (no termination check).
- Allows to derive schemata for pattern/copattern matching.
- Proof that subject reduction holds.

\[ t : A, \quad t \rightarrow t' \implies t' : A \]
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Theorem Regarding Undecidability of Equality

**Theorem**

Assume the following:

- There exists a subset $\text{Stream} \subseteq \mathbb{N}$,
- computable functions $\text{head} : \text{Stream} \to \mathbb{N}$, $\text{tail} : \text{Stream} \to \text{Stream}$,
- a decidable equality $\_ == \_ $ on $\text{Stream}$ which is congruence,
- the possibility to define elements of $\text{Stream}$ by guarded recursion based on primitive recursive functions $f, g : \mathbb{N} \to \mathbb{N}$, such that the standard equalities related to guarded recursion hold.

Then it is not possible to fulfil the following condition:

$$\forall s, s' : \text{Stream}. \text{head } s = \text{head } s' \land \text{tail } s == \text{tail } s' \to s == s'$$

(*)
Remark

*Condition (\(\ast\)) is fulfilled if we have an operation cons : \(\mathbb{N} \rightarrow \text{Stream} \rightarrow \text{Stream}\) preserving equalities s.t.*

\[
\forall s : \text{Stream}. s = \text{cons} (\text{head} s) (\text{tail} s)
\]

So we cannot have a type theory with streams, decidable type checking and decidable equality on streams such that

\[
\forall s. \exists n, s'. s == \text{cons} n s'
\]

as assumed by the codata approach.
Proof of Theorem

- Assume we had the above.
- By
  \[ s \approx n_0 :: n_1 :: n_2 :: \cdots n_k :: s' \]
  we mean the equations using head, tail expressing that \( s \) behaves as
  the stream indicated on the right hand side.
- Define by guarded recursion \( l : \text{Stream} \)
  \[ l \approx 1 :: 1 :: 1 :: \cdots \]
Proof of Theorem

For $e$ code for a Turing machine define by guarded recursion based on primitive recursion functions $f, g$ s.t. if $e$ terminates after $n$ steps and returns result $k$ then

\[ f(e) \approx \begin{cases} 0 : 0 : 0 : \cdots : 0 : l & \text{if } k = 0 \\ n \text{ times} \end{cases} \]

\[ g(e) \approx \begin{cases} 0 : 0 : 0 : \cdots : 0 : l & \text{if } k > 0 \\ 0 : 0 : 0 : \cdots : 0 : l & \text{if } k = 0 \\ n+1 \text{ times} \end{cases} \]
Proof of Theorem

\[ f \ e \ \approx \begin{cases} \text{if } k = 0 & \underbrace{0 :: 0 :: 0 :: \cdots :: 0 :: l} \text{ } n \text{ times} \\ \text{if } k > 0 & \underbrace{0 :: 0 :: 0 :: \cdots :: 0 :: l} \text{ } n+1 \text{ times} \end{cases} \]

- If \( e \) terminates after \( n \) steps with result 0 then

\[ f \ e =\approx g \ e \]

- If \( e \) terminates after \( n \) steps with result \( > 0 \) then

\[ \neg (f \ e =\approx g \ e) \]
Proof of Theorem

- So

\[ \lambda e. (f\ e == g\ e) \]

separates the TM with result 0 from those with result \( > 0 \).

- But these two sets are inseparable.
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Operators for Primitive (Co)Recursion

\[ P_{N,A} : A \rightarrow (N \rightarrow A \rightarrow A) \rightarrow N \rightarrow A \]

\[ P_{N,A} \text{ step}_0 \text{ step}_S \text{ 0 } = \text{ step}_0 \]

\[ P_{N,A} \text{ step}_0 \text{ step}_S \text{ (S n) } = \text{ step}_S \text{ n (P}_{N,A} \text{ step}_0 \text{ step}_S \text{ n)} \]

\[ \text{coP}_{\text{Stream},A} : (A \rightarrow N) \rightarrow (A \rightarrow (\text{Stream} + A)) \rightarrow A \rightarrow \text{Stream} \]

\[ \text{head (coP}_{\text{Stream},A} \text{ step}_{\text{head}} \text{ step}_{\text{tail}} \text{ a)} = \text{ step}_{\text{head}} \text{ a} \]

\[ \text{tail (coP}_{\text{Stream},A} \text{ step}_{\text{head}} \text{ step}_{\text{tail}} \text{ a)} = \]

\[ \text{cases}_{\text{Stream},A,\text{Stream id (coP}_{\text{Stream},A} \text{ step}_{\text{head}} \text{ step}_{\text{tail}} \text{)} (\text{step}_{\text{tail}} \text{ a)} \]
Operators for full/primitive (co)recursion

\[
R_{\mathbb{N},A} : \left( (\mathbb{N} \to A) \to A \right) \to \left( (\mathbb{N} \to A) \to \mathbb{N} \to A \right) \to \mathbb{N} \to A
\]

\[
R_{\mathbb{N},A} \text{ step}_0 \text{ step}_S 0 = \text{ step}_0 \left( R_{\mathbb{N},A} \text{ step}_0 \text{ step}_S \right)
\]

\[
R_{\mathbb{N},A} \text{ step}_0 \text{ step}_S \left( S \ n \right) = \text{ step}_S \left( R_{\mathbb{N},A} \text{ step}_0 \text{ step}_S \right) n
\]

\[
\text{coR}_{\text{Stream}},A : \left( (A \to \text{Stream}) \to A \to \mathbb{N} \right)
\to \left( (A \to \text{Stream}) \to A \to \text{Stream} \right)
\to \text{Stream}
\]

\[
\text{head} \left( \text{coR}_{\text{Stream}},A \text{ step}_\text{head} \text{ step}_\text{tail} \ a \right) = \text{ step}_\text{head} \left( \text{coR}_{\text{Stream}},A \text{ step}_\text{head} \text{ step}_\text{tail} \right) a
\]

\[
\text{tail} \left( \text{coR}_{\text{Stream}},A \text{ step}_\text{head} \text{ step}_\text{tail} \ a \right) = \text{ step}_\text{tail} \left( \text{coR}_{\text{Stream}},A \text{ step}_\text{head} \text{ step}_\text{tail} \right) a
\]
Consider Example from above

\[ f : \mathbb{N} \rightarrow \text{Stream} \]
\[ \text{head} \ (f \ n) \quad = \quad n \]
\[ \text{head} \ (\text{tail} \ (f \ n)) \quad = \quad n \]
\[ \text{tail} \ (\text{tail} \ (f \ 0)) \quad = \quad f \ N \]
\[ \text{tail} \ (\text{tail} \ (f \ (S \ n))) \quad = \quad f \ n \]

This example can be reduced to primitive (co)recursion.

**Step 1:** Following the development of the (co)pattern matching definition, unfold it into simultaneous non-nested (co)pattern matching definitions.
Step 1: Unnesting of Nested (Co)Pattern Matching

We follow the steps in the pattern matching:
We start with

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ \text{head} \ (f \ n) = n \]

\[ \text{tail} \ (f \ n) = ? \]
Copattern matching on tail \((f \ n)\):

\[
f : \mathbb{N} \to \text{Stream}
\]

\[
\text{head} \ (f \ n) = n
\]

\[
\text{head} \ (\text{tail} \ (f \ n)) = n
\]

\[
\text{tail} \ (\text{tail} \ (f \ n)) = ?
\]

corresponds to

\[
f : \mathbb{N} \to \text{Stream}
\]

\[
\text{head} \ (f \ n) = n
\]

\[
\text{tail} \ (f \ n) = g \ n
\]

\[
g : \mathbb{N} \to \text{Stream}
\]

\[
(\text{head} \ (\text{tail} \ (f \ n))) = \) \text{head} \ (g \ n) = n
\]

\[
(\text{tail} \ (\text{tail} \ (f \ n))) = \) \text{tail} \ (g \ n) = ?
\]
Pattern matching on \( \text{tail} (\text{tail} (f \ n)) \):

\[
\begin{align*}
 f &: \mathbb{N} \to \text{Stream} \\
 \text{head} (f \ n) &= n \\
 \text{head} (\text{tail} (f \ n)) &= n \\
 \text{tail} (\text{tail} (f \ 0)) &= f \ N \\
 \text{tail} (\text{tail} (f (S \ n))) &= f \ n
\end{align*}
\]

corresponds to

\[
\begin{align*}
 f &: \mathbb{N} \to \text{Stream} \\
 \text{head} (f \ n) &= n \\
 \text{tail} (f \ n) &= g \ n
\end{align*}
\]

\[
\begin{align*}
 g &: \mathbb{N} \to \text{Stream} \\
 \quad (\text{head} (\text{tail} (f \ n))) &=) \quad \text{head} (g \ n) &= n \\
 \quad (\text{tail} (\text{tail} (f \ n))) &=) \quad \text{tail} (g \ n) &= k \ n
\end{align*}
\]

\[
\begin{align*}
 k &: \mathbb{N} \to \text{Stream} \\
 \quad (\text{tail} (\text{tail} (f \ 0))) &=) \quad k \ 0 &= f \ N \\
 \quad (\text{tail} (\text{tail} (f (S \ n)))) &=) \quad k \ (S \ n) &= f \ n
\end{align*}
\]
Step 2: Reduction to Primitive (Co)recursion

- This can now easily be reduced to full (co)recursion.
- In this example we can reduce it to primitive (co)recursion.
- First combine $f, g$ into one function $f + g$. 
\[ f : \mathbb{N} \rightarrow \text{Stream} \]
\[ f \ n \quad = \quad (f + g) \ (f \ n) \]

\[ (f + g) : (f(\mathbb{N}) + g(\mathbb{N})) \rightarrow \text{Stream} \]
\[ \text{head} \quad ((f + g) \ (f \ n)) \quad = \quad n \]
\[ \text{head} \quad ((f + g) \ (g \ n)) \quad = \quad n \]
\[ \text{tail} \quad ((f + g) \ (f \ n)) \quad = \quad (f + g) \ (g \ n) \]
\[ \text{tail} \quad ((f + g) \ (f \ n)) \quad = \quad k \ n \]

\[ k : \mathbb{N} \rightarrow \text{Stream} \]
\[ k \ 0 \quad = \quad (f + g) \ (f \ N) \]
\[ k \ (S \ n) \quad = \quad (f + g) \ (f \ n) \]
The call of $k$ has result always of the form $(f + g)(\text{fbf } n))$. So we can replace the recursive call $k \ n$ by $(f + g)(f (k' \ n))$. 
\[ f : \mathbb{N} \rightarrow \text{Stream} \]
\[ f \ n \quad = \quad (f + g) \ (\overline{f} \ n) \]

\[(f + g) : (\overline{f}(\mathbb{N}) + g(\mathbb{N})) \rightarrow \text{Stream} \]

\[ \text{head} \ ((f + g) \ (\overline{f} \ n)) \quad = \quad n \]
\[ \text{head} \ ((f + g) \ (g \ n)) \quad = \quad n \]
\[ \text{tail} \ ((f + g) \ (\overline{f} \ n)) \quad = \quad (f + g) \ (g \ n) \]
\[ \text{tail} \ ((f + g) \ (\overline{f} \ n)) \quad = \quad (f + g) \ (\overline{f} \ (k' \ n)) \]

\[ k' : \mathbb{N} \rightarrow \mathbb{N} \]
\[ k \ 0 \quad = \quad N \]
\[ k \ (S \ n) \quad = \quad n \]
Unfolding of the Pattern Matchings

- $(f + g)$ can be defined by primitive corecursion.
- $k'$ can be defined by primitive recursion.
\( f : \mathbb{N} \to \text{Stream} \)
\[
f n = (f + g) (f \ n)
\]

\( (f + g) : (f(\mathbb{N}) + g(\mathbb{N})) \to \text{Stream} \)
\[
(f + g) = 
\text{coP}_{\text{Stream}, (f(\mathbb{N}) + g(\mathbb{N}))} \ (\lambda x. \text{case}_r(x) \ \text{of} \ 
  (f \ n) \to n \\
  (g \ n) \to n)
\]
\[
(\lambda x. \text{case}_r(x) \ \text{of} \ 
  (f \ n) \to g \ n \\
  (g \ n) \to f' (k' \ n))
\]

\( k' : \mathbb{N} \to \mathbb{N} \)
\[
k' = \text{P}_{\mathbb{N}, \mathbb{N}} \ N (\lambda n, ih.n)
\]
The case distinction can be trivially replaced by the case distinction operator.
\[ f : \mathbb{N} \rightarrow \text{Stream} \]
\[ f \ n = (f + g) (\overline{f} \ n) \]

\[(f + g) : (\overline{f}(\mathbb{N}) + g(\mathbb{N})) \rightarrow \text{Stream} \]
\[(f + g) = \]
\[\text{coP}_{\text{Stream}, \overline{f}(\mathbb{N}) + g(\mathbb{N})} \]
\[\begin{cases}
(\text{case}_{\overline{f}(\mathbb{N}) + g(\mathbb{N})} \text{id} \text{id}) \\
(\text{case}_{\overline{f}(\mathbb{N}) + g(\mathbb{N})} g \ (f \circ k'))
\end{cases} \]

\[ k' : \mathbb{N} \rightarrow \mathbb{N} \]
\[ k' = P_{\mathbb{N}, \mathbb{N}} \mathbb{N} (\lambda n, \text{ih}.n) \]
From Codata to Coalgebras

Algebras and Coalgebras

Patterns and Copatterns

Codata types and Decidable Equality

Reduction of Mixed Pattern/Copattern Matching to Operators

Conclusion
Conclusion

- Codata types make the assumption

\[ \forall s : \text{Stream}. \exists n, s'. s = \text{cons } n \ s' \]

which cannot be combined with a decidable equality.

- In general Codata types cause problems such as subject reduction.

- Solution:
  - Coalgebra are determined by their elimination rule.
  - Introduction rule corresponds to copattern matching.

- Solves problem of subject reduction.
One can reduce certain cases of recursive nested (co)pattern matching to primitive (co)recursion.

- Systematic treatment needs still to be done.
- Cases which can be reduced should be those to be accepted by a termination checker.
- If the reduction succeeds we get a normalising version (by Mendler and Geuvers).
- Therefore a termination checked version of the calculus is normalising.