The extended predicative Mahlo Universe and the need for partial proofs and partial objects in type theory.

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Inductive-Recursive Definition of Universes

mutual

data $\mathrm{U}_0 : \mathrm{Set}$ where

$\widehat{\mathbb{N}} \quad : \quad \mathrm{U}_0$

$\widehat{\mathrm{W}} \quad : \quad (x : \mathrm{U}_0) \to (\mathrm{T}_0\ x \to \mathrm{U}_0) \to \mathrm{U}_0$

$\cdots$

$\mathrm{T}_0 : \mathrm{U}_0 \to \mathrm{Set}$

$\mathrm{T}_0\ \widehat{\mathbb{N}} \quad = \quad \mathbb{N}$

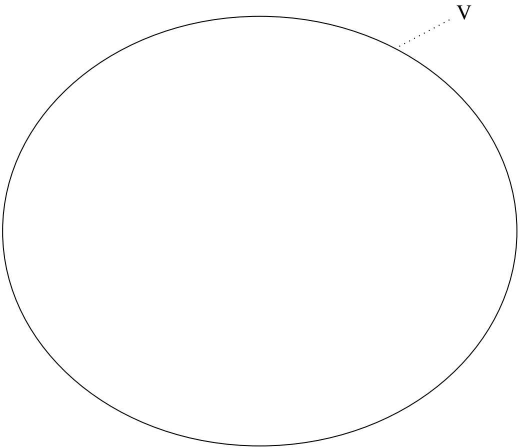
$\mathrm{T}_0\ (\widehat{\mathrm{W}}\ a\ b) \quad = \quad \mathrm{W} x : \mathrm{T}_0\ a . \mathrm{T}_0\ (b\ x)$

$\cdots$

Universe Operator and Super Universe (Palmgren)

- Define
$$\mathrm{Fam}(\mathrm{Set}) = \Sigma X : \mathrm{Set}.X \to \mathrm{Set}$$
In rules $\mathrm{Fam}(\mathrm{Set})$ is avoided by Currying.
- Define
$$\mathrm{U}^+ : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$$
such that $\mathrm{U}^+\ A$ is a **universe containing (codes for) A**.
- Let a **super universe** be a universe closed under U^+.

Illustration of the Super Universe



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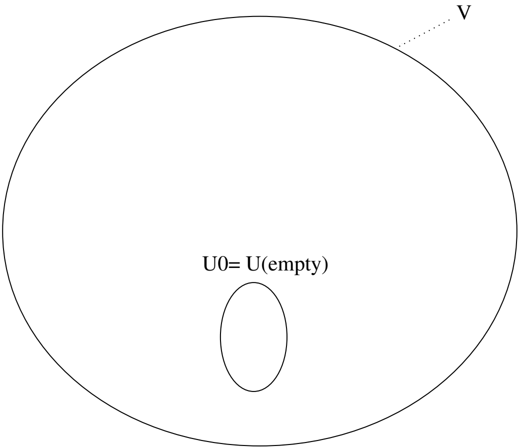

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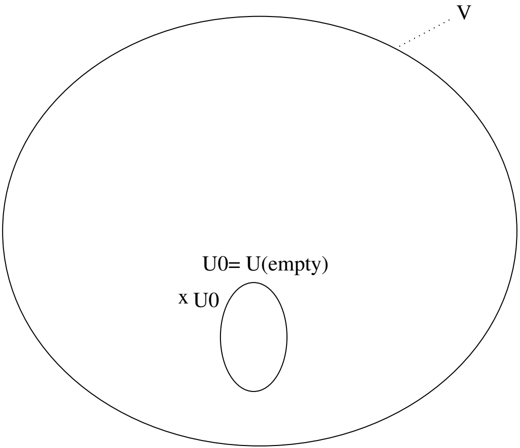

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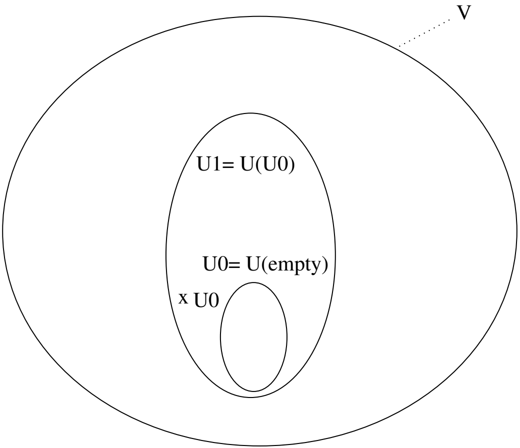

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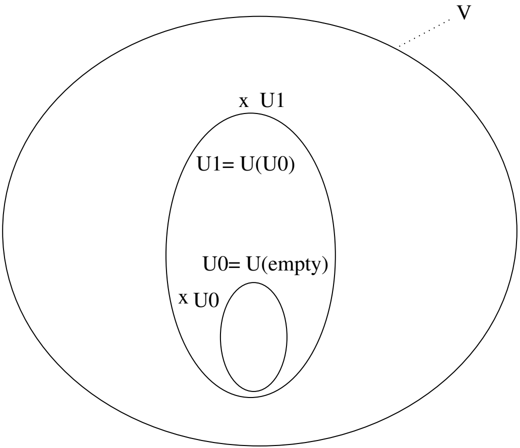

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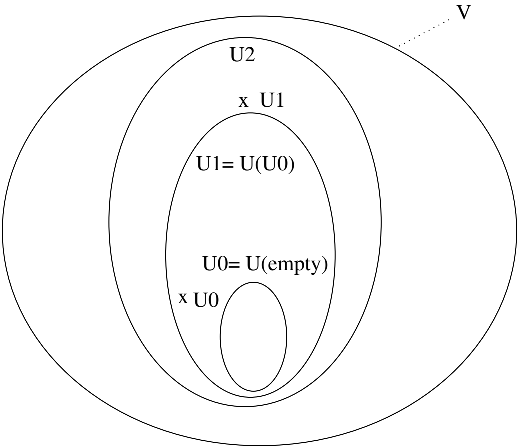

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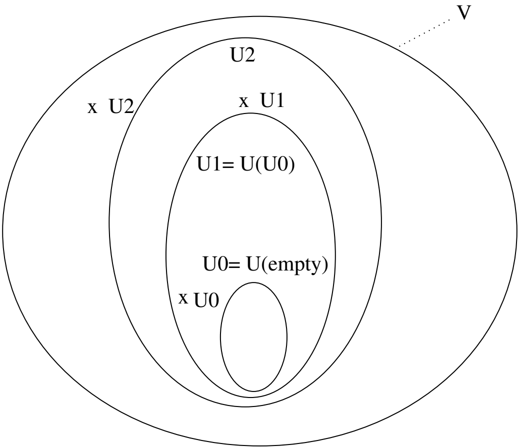

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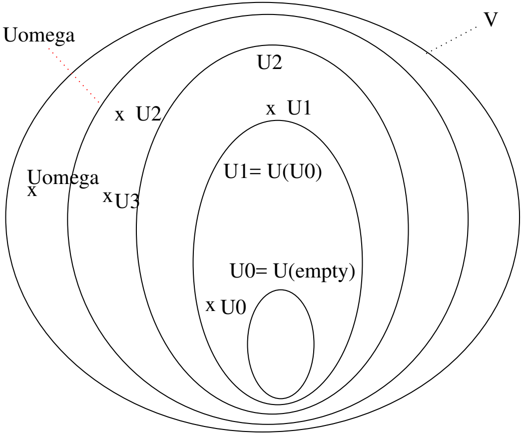


Super Universe Operator (Palmgren)

- Define the super universe operator

$$\mathrm{SU} : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$$

  where $\mathrm{SU}\ A$ is a super universe containing A.
- Let a **super-super universe** be a universe closed under SU.

External Mahlo Universe

- Generalise the above to allow formation of universes closed under arbitrary operators:
  - If $f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$ then

$$\mathrm{U}_f : \mathrm{Fam}(\mathrm{Set})$$

    is a universe closed under f.
- The **external Mahlo universe** is the type theory formalising the existence of U_f for any such f.

Internal Mahlo Universe

- The **internal Mahlo universe** V is a universe internalising closure under $\lambda f.\mathrm{U}_f$: If

$$f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$$

then

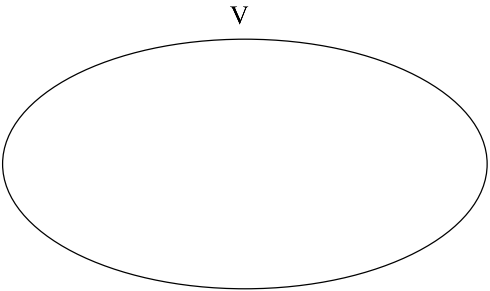
$$\widehat{\mathrm{U}}_f : \mathrm{Fam}(\mathrm{V})$$

is a family of codes for a subuniverse

$$\mathrm{U}_f : \mathrm{Fam}(\mathrm{Set})$$

of V closed under f

Illustration of the Mahlo Universe



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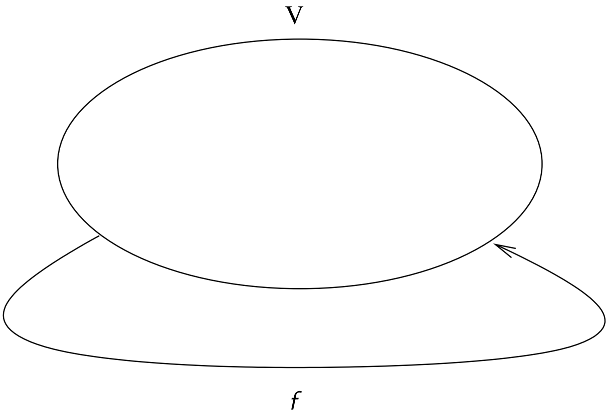

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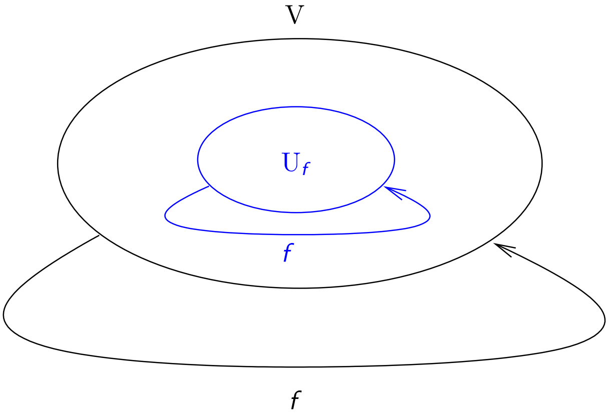

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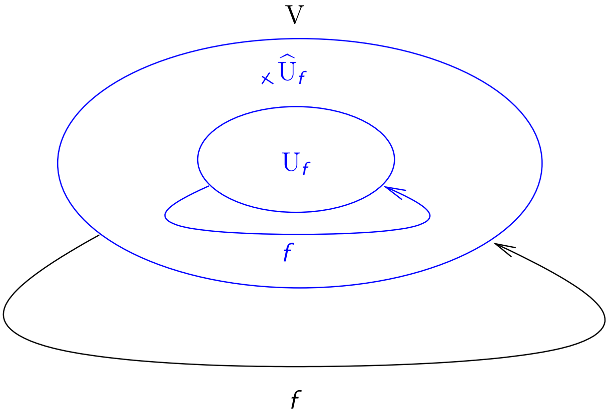


Problems of Mahlo Universe

- Constructor
  $$\widehat{\mathrm{U}} : (\mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})) \to \mathrm{V}$$
  refers to
  - the set of total functions
    $$\mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$$
  - which depends on the totality of V.
- So the reason for adding $\widehat{\mathrm{U}}$ depends on the totality of V.
  - Less problematic since we don't have elimination rules for V.
  - So functions $\mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$ can only make use of introduction rules for forming elements of V.
  - However, this idea hasn't been transformed yet into a formal model of the Mahlo universe.

Predicative Solution to the Mahlo Universe

- For defining U_f, only the restriction of f to $\mathrm{Fam}(\mathrm{U}_f)$ is required to be total.
  - Only local knowledge of V is needed.
  - Adding $\widehat{\mathrm{U}}_f$ to V does not affect the reason for adding it.

Idea for an Extended Predicative Mahlo Universe

- Idea: For partial functions f define a subuniverse

$$\mathrm{Pre}\ f\ \mathrm{V}$$

  of V closed under f as long as it stays in V.
- If $\mathrm{Pre}\ f\ \mathrm{V}$ is closed under f, then we add a code $\widehat{\mathrm{U}}_f$ for $\mathrm{Pre}\ f\ \mathrm{V}$ to V.
- Therefore reason for adding $\widehat{\mathrm{U}}_f$ doesn't depend on totality of V, V is **predicative**.
- Requires that we have the notion of a **partial function** f.

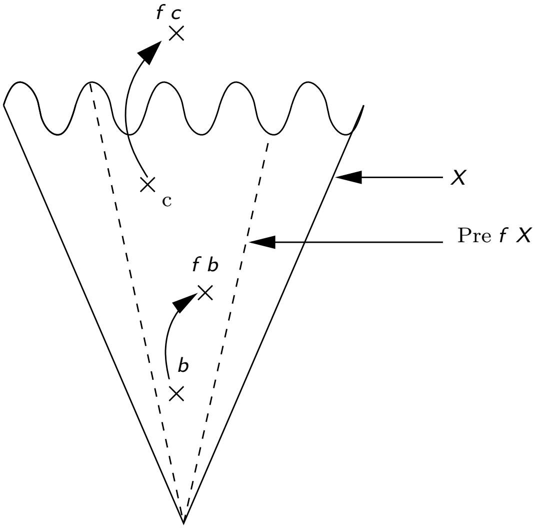
Explicit Mathematics (EM)

- Problem: In MLTT we have no access to the set of partial functions (“potential programs”, collection of terms of our language).
- In Feferman’s explicit mathematics (EM) this exist.
- We will work in EM, but use syntax borrowed from type theory,
  - however write $a \in B$ instead of $a : B$.

Basics of EM

- EM more Russell-style, therefore we can have
  - $\mathrm{V} \in \mathrm{Set}$,
  - $\mathrm{V} \subset \mathrm{Set}$,
  - no need to distinguish between $\widehat{\mathrm{U}}$ and U.
- We can encode $\mathrm{Fam}(\mathrm{V})$ into V, therefore need only to consider functions
$$f : \mathrm{V} \to \mathrm{V}$$
- We define now f, $X \in \mathrm{Set}$, $X \subseteq \mathrm{Set}$
$$\mathrm{Pre}\ f\ X \in \mathrm{Set} \qquad \mathrm{Pre}\ f\ X \subseteq X$$

Pre f X



Closure of Pre f X

- $\mathrm{Pre}\ f\ X$ is closed under universe constructions, if result is in X.
- Closure under Σ (called join in EM):

$$\forall a \in \mathrm{Fam}(\mathrm{Pre}\ f\ X).\Sigma\ a \in X \to \Sigma\ a \in \mathrm{Pre}\ f\ X$$

- $\mathrm{Pre}\ f\ X$ is closed under f, if result is in X:

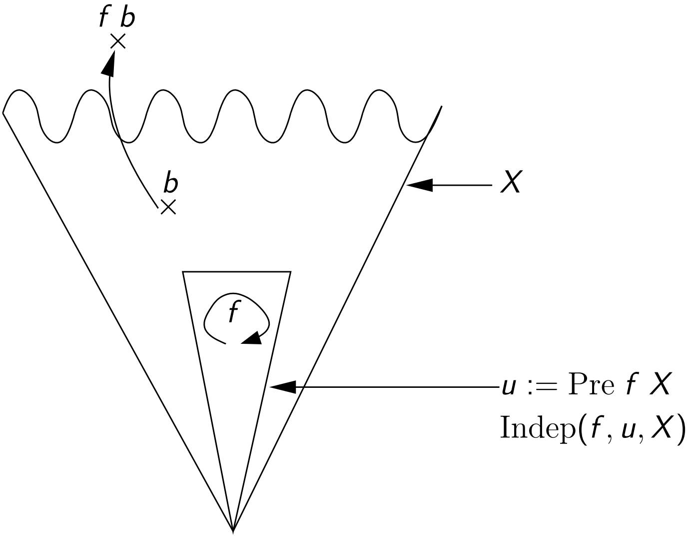
$$\forall a \in \mathrm{Pre}\ f\ X.\ f\ a \in X \to f\ a \in \mathrm{Pre}\ f\ X$$

Independence of Pre f X

- If, whenever a universe construction or f is applied to elements of $\mathrm{Pre}\ f\ X$ we get elements in X, then $\mathrm{Pre}\ f\ X$ is independent of future extensions of X.

$$\begin{aligned}\mathrm{Indep}(f, \mathrm{Pre}\ f\ X, X) := & (\forall a \in \mathrm{Fam}(\mathrm{Pre}\ f\ X).\Sigma\ a \in X)\\ & \wedge\ \cdots\\ & \wedge f[\mathrm{Pre}\ f\ X] \subseteq X\end{aligned}$$

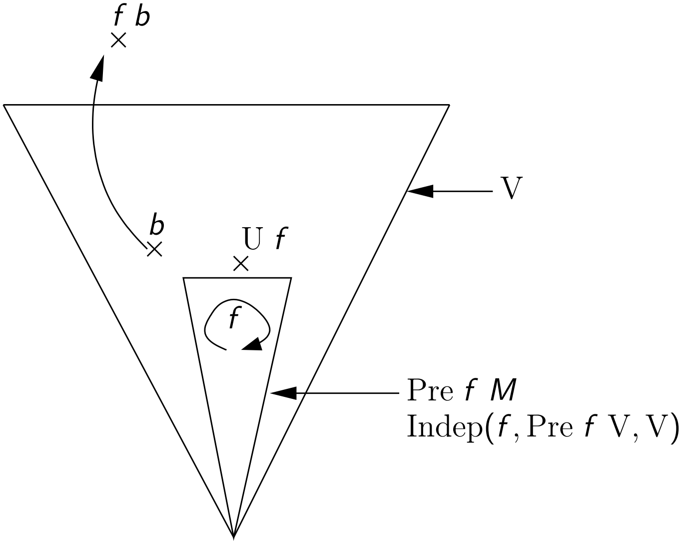
Indep X f



Introduction Rule for V

- $\forall f.\, \mathrm{Indep}(f, \mathrm{Pre}\ f\ \mathrm{V}, \mathrm{V}) \to (\mathrm{U}_f \in \mathrm{Set}$
  $\wedge\ \mathrm{U}_f =_{\mathrm{ext}} \mathrm{Pre}\ f\ \mathrm{V}$
  $\wedge\ \mathrm{U}_f \in \mathrm{V})$.
- V admits an **elimination rule** expressing that V is the smallest universe closed under universe constructions and introduction of U_f.

Introduction Rule for V



Interpretation of Axiomatic Mahlo

- It easily follows:

$$\forall f \in \mathrm{V} \to \mathrm{V}.\, \mathrm{Indep}(f, \mathrm{Pre}\ f\ \mathrm{V}, \mathrm{V})$$

therefore

$$\forall f \in \mathrm{V} \to \mathrm{V}.\, \mathrm{U}_f \in \mathrm{V}\ \wedge\ \mathrm{Univ}(f)\ \wedge\ f \in \mathrm{U}_f \to \mathrm{U}_f$$

- So V closed under axiomatic Mahlo constructions.
- Therefore extended predicative Mahlo has at least strength of axiomatic Mahlo.
- We are currently working out the details of the model of the extended predicative Mahlo universe.

Steps towards the Mahlo Universe

Extended Predicative Mahlo

Partial Proofs and Objects

Partial Functions

- $s \in t$ is undecidable (since s can have infinitely many reductions).
- So in

$$s \in t$$

  s is only a realiser not a proof of t.
- The proof that $s \in t$ is given by the derivation that $s \in t$.
- Can be made explicit by having proof objects:

$$p : s \in t$$

- In Martin-Löf Type theory p and s are encoded into one object.
  - Causes problems because there different proofs of $s \in t$ result in different elements of t which correspond to the same realiser.

First Steps towards defining Extended Predicative Mahlo in Agda

```
data Term : Set where
  nat zero m : Term
  suc        : Term → Term
  ap pu      : Term → Term → Term

data _~>_ : (s t : Term) → Set where

data _∈Nat : Term → Set where
  zeroproof : zero ∈Nat
  sucproof  : (t : Term) → t ∈Nat → suc t ∈Nat
  eqproof   : (s t : Term) → s ~> t → t ∈Nat → s ∈Nat
```

First Steps towards defining Extended Predicative Mahlo in Agda

```
data _∈Set : Term → Set where
  natproof : nat ∈Set
  puproof  : (a f : Term) → pu a f ∈Set
  mproof   : m ∈Set
  eqproof  : (s t : Term) → s ~> t → t ∈Set → s ∈Set
```

First Steps towards defining Extended Predicative Mahlo in Agda

```
mutual
  data _∈PU[_,_] : (x a f : Term) → Set where
    aproof   : {a f : Term} → a ∈M → a ∈PU[ a , f ]
    fproof   : {a f x : Term} → x ∈PU[ a , f ] → ap f x ∈M
               → ap f x ∈PU[ a , f ]
    eqproof  : {a f x : Term} → (s t : Term) → s ~> t
               → t ∈PU[ a , f ] → s ∈PU[ a , f ]

  data _∈M : Term → Set where
    u : (a f : Term)
        → a ∈M
        → ((x : Term) → x ∈PU[ a , f ] → ap f x ∈M)
        → pu a f ∈M
    eq  : (s t : Term) → s ~> t → t ∈M → s ∈M
```

First Steps towards defining Extended Predicative Mahlo in Agda

```
PU2M : (a f x : Term) → x ∈PU[ a , f ] → x ∈M
PU2M a f .a (aproof am)          = am
PU2M a f (ap .f y) (fproof ypu p) = p
PU2M a f s (eqproof s t st tpu)   = eq s t st (PU2M a f t tpu)
```

```
M2set : (t : Term) → t ∈M → t ∈Set
M2set .(pu a f) (u a f m′ indep) = puproof a f
M2set s (eq s t st tm)           = eqproof s t st (M2set t tm)
```

```
PU2Set : (a f x : Term) → x ∈PU[ a , f ] → x ∈Set
PU2Set a f x ispu = M2set x (PU2M a f x ispu)
```

First Steps towards defining Extended Predicative Mahlo in Agda

```
_η_ : (s t : Term) → Set
s η nat      = s ∈Nat
s η m        = s ∈M
s η pu a  f  = s ∈PU[ a , f ]
s η _        = ⊥
```

Conclusion

- Predicative presentation of the Mahlo universe based on partial functions.
- Requires access to the collection of all partial objects.
- Separation of realisers and proofs.