Pattern and Copattern matching

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Iteration, Recursion, Induction

Coiteration, Corecursion

Bisimilarity and Coinduction

Proofs by Coinduction of Bisimilarity in Transition Systems

Mixed Patterns and Copatterns

Unnesting of Pattern/Copattern Matching
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Unnesting of Pattern/Copattern Matching
\( \mathbb{N} \) as an Initial Algebra

- \( \mathbb{N} \) is initial algebra of the functor \( F(X) = 1 + X \)

\[
\begin{align*}
F(\mathbb{N}) &= 1 + \mathbb{N} \\
F(g) &= 1 + g \\
F(A) &= 1 + A
\end{align*}
\]

\[ F(N) = 1 + N \quad \xrightarrow{0 + S} \quad N \]

\[ F(g) = 1 + g \quad \xrightarrow{\exists!\ g} \quad \mathbb{N} \]

\[ F(A) = 1 + A \quad \xrightarrow{f'} \quad A \]

\( f' \) can be decomposed as \( f' = a + f \)
Unique Iteration

Unique existence of $g$ means **unique iteration**:

Given $a : A$ and $f : A \rightarrow A$ there exists a unique

$$g : \mathbb{N} \rightarrow A$$

$$g \ 0 \ = \ a$$

$$g \ (S \ n) \ = \ f \ (g \ n)$$

i.e

$$g \ (S^n \ 0) \ = \ f^n \ a$$
From the principle of unique iteration we can prove the principle of unique (primitive) recursion:

Given $a : A$ and $f : \mathbb{N} \to A \to A$ there exists a unique

$$
g : \mathbb{N} \to A$$

$$
g 0 = a$$

$$
g (S n) = f n (g n)$$
Induction

- From the principle of unique iteration we can prove the principle of induction:

  Assume $A : \mathbb{N} \to \text{Set}$, $a : A \ 0$ and $f : (n : \mathbb{N}) \to A \ n \to A \ (S \ n)$

  There exists a unique

  $$g : (n : \mathbb{N}) \to A \ n$$

  $$g \ 0 = a$$

  $$g \ (S \ n) = f \ n \ (g \ n)$$

- Using induction we can prove that if we have two solutions for a iteration or recursion principle, they are pointwise equal, i.e. uniqueness of iteration and recursion.
The above means that we can define

\[
g : (n : \mathbb{N}) \rightarrow A \ n
\]

\[
g \ 0 = a \ \text{for some} \ a : A
\]

\[
g \ (S \ n) = a' \ \text{for some} \ a' : A \ \text{depending on} \ n
\]

where in the second case we can use the **recursion hypothesis** or **induction hypothesis** \( g \ n \).

This means we can define \( g \ n \) by **pattern matching** on \( n : \mathbb{N} \).
Theorem

Assume \( \mathbb{N} : \text{Set}, \, 0 : \mathbb{N}, \, S : \mathbb{N} \rightarrow \mathbb{N} \).

The following are equivalent:

- The principle of unique iteration.
- The principle of unique recursion.
- The principle of unique induction.
- The principle of induction.
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Unnesting of Pattern/Copattern Matching
Streams as a Final Coalgebra

- Dual of $\oplus$ is $\times$, so we use for clarity a functor using product rather than disjoint union:

- **Stream** is the final coalgebra of $F(X) = \mathbb{N} \times X$

$$
\begin{align*}
X & \xrightarrow{f} \mathbb{N} \times X = F(X) \\
\exists! g & \text{ s.t. } \text{id} \times g = F(g) \\
\text{Stream} & \xrightarrow{\text{head} \times \text{tail}} \mathbb{N} \times \text{Stream} = F(\text{Stream})
\end{align*}
$$

- We can decompose $f$ as

$$
f = f_0 \times f_1$$
Unique Coiteration

This corresponds to the principle of unique coiteration:
There exists a unique

\[ g : A \rightarrow \text{Stream} \]
\[ \text{head} (g \ x) = f_0 \ x \]
\[ \text{tail} (g \ x) = g \ (f_1 \ x) \]
Unique Coiteration

- We had:

  \[ \text{head} \left( g \ x \right) = f_0 \ x \]
  \[ \text{tail} \left( g \ x \right) = g \left( f_1 \ x \right) \]

- By choosing \( f_0, f_1 \) we can define \( g : X \rightarrow \text{Stream} \) s.t.

  \[ \text{head} \left( g \ x \right) = n \quad \text{for some } n : \mathbb{N} \text{ depending on } x \]
  \[ \text{tail} \left( g \ x \right) = g \ x' \quad \text{for some } x' : X \text{ depending on } x \]
From unique coiteration we can derive **unique corecursion**: There exists a unique

\[ g : A \to \text{Stream} \]

- head \( (g \ x) = n \) for some \( n : \mathbb{N} \) depending on \( x \)
- tail \( (g \ x) = g \ x' \) for some \( x' : X \) depending on \( x \)
  - or
  - \( = s \) for some \( s : \text{Stream} \) depending on \( x \)

This means we can define \( g \ x \) by **copattern matching**
Examples

We can define

\[
\text{cons} : (\mathbb{N} \times \text{Stream}) \rightarrow \text{Stream}
\]
\[
\text{head} (\text{cons}(n, s)) = n
\]
\[
\text{tail} (\text{cons}(n, s)) = s
\]

Note: \text{cons} not primitive but \textbf{defined} by corecursion

\[
\text{inc} : \mathbb{N} \rightarrow \text{Stream}
\]
\[
\text{head} (\text{inc } n) = n
\]
\[
\text{tail} (\text{inc } n) = \text{inc } (n + 1)
\]
Examples

\[
\begin{align*}
\text{inc}' & : \mathbb{N} \rightarrow \text{Stream} \\
\text{head} \ (\text{inc}'(n)) & = n \\
\text{tail} \ (\text{inc}'(n)) & = \text{inc}''(n + 1)
\end{align*}
\]

\[
\begin{align*}
\text{inc}'' & : \mathbb{N} \rightarrow \text{Stream} \\
\text{head} \ (\text{inc}''(n)) & = n \\
\text{tail} \ (\text{inc}''(n)) & = \text{inc}'(n + 1)
\end{align*}
\]
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Unnesting of Pattern/Copattern Matching
Bisimilarity

- Bisimilarity $\sim$ on Streams is an **indexed final coalgebra**.
- Consider the category $\text{Set}^{\text{Stream} \times \text{Stream}}$ of binary relations

\[ \varphi : \text{Stream} \times \text{Stream} \to \text{Set} \]

- Let

\[ F^\sim : \text{Set}^{\text{Stream} \times \text{Stream}} \to \text{Set}^{\text{Stream} \times \text{Stream}} \]

\[ F^\sim(\varphi, (s, s')) = (\text{head } s = \text{head } s') \times \varphi (\text{tail } s, \text{tail } s') \]
That $\sim$ is a $F^{\sim}$ coalgebra means there exist

$$\text{elim}_{\sim} : (s, s' : \text{Stream}) \rightarrow s \sim s' \rightarrow (\text{head } s = \text{head } s') \times (\text{tail } s \sim \text{tail } s')$$

i.e.

$$s \sim s' \rightarrow (\text{head } s = \text{head } s') \land ((\text{tail } s) \sim (\text{tail } s'))$$

Let $\text{elim}_{\sim}^0$ and $\text{elim}_{\sim}^1$ the two components of $\text{elim}_{\sim}$,

$$\text{elim}_{\sim}^0 : (s, s' : \text{Stream}) \rightarrow s \sim s' \rightarrow \text{head } s = \text{head } s'$$
$$\text{elim}_{\sim}^1 : (s, s' : \text{Stream}) \rightarrow s \sim s' \rightarrow \text{tail } s \sim \text{tail } s'$$

and hide the first two arguments of $\text{elim}_{\sim}^i$. 
Bisimilarity

- That $\sim$ is a final $F^{\sim}$-coalgebra means that it is the largest such relation:

$$\varphi(s, s') \xrightarrow{f} \text{head } s = \text{head } s' \land \varphi(\text{tail } s, \text{tail } s')$$

$$\exists! g \quad \text{id} \land g$$

$$s \sim s' \xrightarrow{\text{elim} \sim} \text{head } s = \text{head } s' \land (\text{tail } s) \sim (\text{tail } s')$$

- This means that

$$\forall s, s'. \varphi(s, s') \rightarrow \text{head } s = \text{head } s' \land \varphi(\text{tail } s, \text{tail } s')$$

then

$$\forall s, s'. \varphi(s, s') \rightarrow s \sim s'$$
Bisimilarity

- So we have

\[ s \sim s' \rightarrow \text{head } s = \text{head } s' \land (\text{tail } s) \sim (\text{tail } s') \]

and if

\[ \forall s, s'. \varphi (s, s') \rightarrow \text{head } s = \text{head } s' \land \varphi (\text{tail } s, \text{tail } s') \]

then

\[ \forall s, s'. \varphi (s, s') \rightarrow s \sim s' \]
Corecursive Proof of Bisimilarity

- Because $\sim$ is a final coalgebra we can compute proofs of it by corecursion:
- We can define

$$f : (s, s' : \text{Stream}) \rightarrow \varphi s s' \rightarrow s \sim s'$$

$$\text{elim}_0^\sim (f \ s \ s' \ x) = \text{an element of head } s = \text{head } s'$$

$$\text{elim}_0^\sim (f \ s \ s' \ x) = \text{an element of (tail } s) \sim (\text{tail } s')$$

where in the last line we can use

- either a proof of tail $s \sim$ tail $s'$ defined before
- or use the corecursion hypothesis $f (\text{tail } s) (\text{tail } s') x'$ for some $x' : \varphi (\text{tail } s) (\text{tail } s')$
The following are equivalent

▶ The principle of unique coiteration.
▶ The principle of unique corecursion.
▶ The principle of iteration together with the principle that bisimilarity $\sim$ implies equality

$$\forall s, s' : \text{Stream}. s \sim s' \rightarrow s = s'$$

Because of the possibility of defining elements of $s \sim s'$ the latter can be considered as a **principle of coinduction**.
Let $\varphi : \text{Stream} \to \text{Stream} \to \text{Set}$.

We can prove
\[
\forall s, s' : \text{Stream}. \varphi s s' \to s = s'
\]
by showing
\[
\forall s, s' : \text{Stream}. \varphi s s' \to \text{head } s = \text{head } s'
\]
\[
\forall s, s' : \text{Stream}. \varphi s s' \to \text{tail } s = \text{tail } s'
\]

where for proving $\text{tail } s = \text{tail } s'$ we can use the coinduction hypothesis that $\varphi (\text{tail } s) (\text{tail } s')$ implies $\text{tail } s = \text{tail } s'$. 
Indexed Coinduction

Instead of defining $\varphi$ as a predicate $\text{Stream} \to \text{Stream} \to \text{Set}$ we can assume

$$A : \text{Set}$$
$$s, t : A \to \text{Stream}$$

and define

$$\varphi \ s' \ t' = (a : A) \times (s' = s \ a) \times (t' = t \ a)$$

Coinduction of $\varphi$ becomes then the principle of indexed coinduction (see next slide)
Indexed Coinduction

- Assume

\[ A \rightarrow Set \]
\[ s_0, s_1 : A \rightarrow Stream \]

- We can prove

\[ \forall a : A. s_0 a = s_1 a \]

by showing

\[ \forall a : A. \text{head} (s a) = \text{head} (t a) \]
\[ \forall a : A. \text{tail} (s a) = \text{tail} (t a) \]

where for proving \( \text{tail} (s a) = \text{tail} (t a) \) we can use that \( \text{tail} (s a) = s a' \) and \( \text{tail} (t a) = t a' \) and therefore by \textbf{coinduction-hypothesis} \( s a' = t a' \).
Example Proof by Coinduction

- Remember

\[
\begin{align*}
\text{inc} & : \mathbb{N} \rightarrow \text{Stream} \\
\text{head}(\text{inc } n) & = n \\
\text{tail} \ (\text{inc } n) & = \text{inc} \ (n + 1) \\
\text{inc}' & : \mathbb{N} \rightarrow \text{Stream} \\
\text{head}(\text{inc}'(n)) & = n \\
\text{tail} \ (\text{inc}'(n)) & = \text{inc}''(n + 1) \\
\text{inc}'' & : \mathbb{N} \rightarrow \text{Stream} \\
\text{head}(\text{inc}''(n)) & = n \\
\text{tail} \ (\text{inc}''(n)) & = \text{inc}'(n + 1)
\end{align*}
\]
Example Proof by Coinduction

- We show

\[ \forall n \in \mathbb{N}. \text{inc}' n = \text{inc} n \land \text{inc}'' n = \text{inc} n \]

- Formally we would use in the above

\[ A = \mathbb{N} + \mathbb{N} \]
\[ s \ (\text{inl} \ n) = \text{inc}' n \]
\[ s \ (\text{inr} \ n) = \text{inc}'' n \]
\[ t \ (\text{inl} \ n) = \text{inc} n \]
\[ t \ (\text{inr} \ n) = \text{inc} n \]

and show

\[ \forall a : A. s \ a = t \ a \]
Example Proof by Coinduction

Proof of

∀n ∈ ℕ. \text{inc}' n = \text{inc} n ∧ \text{inc}'' n = \text{inc} n

Assume n : ℕ.

head (\text{inc}' n) = n = head (\text{inc} n)
head (\text{inc}'' n) = n = head (\text{inc} n)

\text{tail} (\text{inc}' n) = \text{inc}'' (n + 1) \overset{\text{co-IH}}{=} \text{inc} (n + 1) = \text{tail} (\text{inc} n)
\text{tail} (\text{inc}'' n) = \text{inc}' (n + 1) \overset{\text{co-IH}}{=} \text{inc} (n + 1) = \text{tail} (\text{inc} n)
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Unnesting of Pattern/Copattern Matching
Consider the following (unlabelled) transition system:

\[
p \sim q \rightarrow (\forall p'. p \rightarrow p' \\
\rightarrow \exists q'. q \rightarrow q' \land p' \sim q') \\
\land \cdots \text{ symmetric case} \cdots \}
\]
Proof using the Definition of $\sim$

- We show $p \sim q \land p \sim r$ by coinduction:

  - **Coinduction step for** $p \sim q$:
    - Assume $p \xrightarrow{} p'$. Then $p' = p$.
    - We have $q \xrightarrow{} r$ and by co-IH $p \sim r$.
    - Assume $q \xrightarrow{} q'$. Then $q' = r$.
    - We have $p \xrightarrow{} p$ and by co-IH $p \sim r$.

  - **Coinduction step for** $p \sim r$:
    - Assume $p \xrightarrow{} p'$. Then $p' = p$.
    - We have $r \xrightarrow{} q$ and by co-IH $p \sim q$.
    - Assume $r \xrightarrow{} r'$. Then $r' = q$.
    - We have $p \xrightarrow{} p$ and by co-IH $p \sim q$. 
The standard argument for showing $p \sim q \land p \sim r$ is as follows:

Define a relation $\varphi$ on states by

$$\varphi(p', q') \iff p' = p \land (q' = q \lor q' = r)$$

Show $\varphi$ is a simulation:

$$\forall p, p', q. \varphi(p, q) \land p \rightarrow p' \Rightarrow \exists q'. q \rightarrow q' \land \varphi(p', q')$$

$$\forall p, q, q'. \varphi(p, q) \land q \rightarrow q' \Rightarrow \exists p'. p \rightarrow p' \land \varphi(p', q')$$
Comparison with Proofs by Induction

- We can compare both proofs to proofs by induction on natural number. Consider a proof of
\[ \forall n, m, k. n + (m + k) = (n + m) + k \]
- The traditional proof would correspond to defining a relation
\[ R(k) \iff \forall n, m. n + (m + k) = (n + m) + k \]
  and showing
\[ R(0) \land \forall n. R(n) \to R(S(n)) \]
- Although this argument and the standard inductive proof using the induction hypothesis are equivalent, the standard inductive proof is more convenient and easier to follow.
- We hope that proofs by coinduction will similarly be easier if we do it by referring to the coinduction hypothesis.
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Mixed Patterns and Copatterns

Unnesting of Pattern/Copattern Matching
Nested Pattern Matching

- Course of Value primitive recursion allows deep pattern matching.
  E.g. we can define the Fibonacci numbers

\[
\begin{align*}
\text{fib} & : \mathbb{N} \to \mathbb{N} \\
\text{fib} \ 0 & = 1 \\
\text{fib} \ (S \ 0) & = 1 \\
\text{fib} \ (S \ (S \ n)) & = \text{fib} \ n + \text{fib} \ (S \ n)
\end{align*}
\]

- We can now even mix pattern and copattern matching.
We can define new functions by patterns and copatterns. Example:

Define stream:\n\[ f \ n = \ n, n, n - 1, n - 1, \ldots 0, 0, N, N, N - 1, N - 1, \ldots 0, 0, N, N, N - 1, N - 1, \ldots \]
Example Mixed Pattern/Copattern Matching

\[ f(n) = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f = ? \]
Example Mixed Pattern/Copattern Matching

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

Copattern matching on \( f : \mathbb{N} \rightarrow \text{Stream} \):

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f \ n = ? \]

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Example Mixed Pattern/Copattern Matching

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f \ n = ? \]

**Copattern matching** on \( f \ n : \text{Stream} \):

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ \text{head} \ (f \ n) = ? \]

\[ \text{tail} \ (f \ n) = ? \]
Example Mixed Pattern/Copattern Matching

\[
f(n) = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots
\]

\[
f : \mathbb{N} \to \text{Stream}
\]

\[
f(n) = ?
\]

Solve first case, copattern match on second case:

\[
f : \mathbb{N} \to \text{Stream}
\]

\[
\text{head} \ (f(n)) = n
\]

\[
\text{head} \ (\text{tail} \ (f(n))) = ?
\]

\[
\text{tail} \ (\text{tail} \ (f(n))) = ?
\]
Example Mixed Pattern/Copattern Matching

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f \ n = ? \]

Solve second line, pattern match on \( n \)

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ \text{head} \ (f \ n) \ = \ n \]

\[ \text{head} \ (\text{tail} \ (f \ n)) \ = \ n \]

\[ \text{tail} \ (\text{tail} \ (f \ 0)) \ = \ ? \]

\[ \text{tail} \ (\text{tail} \ (f \ (S \ n))) \ = \ ? \]
Example Mixed Pattern/Copattern Matching

\[ f \ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1, \ldots \]

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ f \ n = ? \]

Solve remaining cases

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[ \text{head} \ (f \ n) = n \]

\[ \text{head} \ (\text{tail} \ (f \ n)) = n \]

\[ \text{tail} \ (\text{tail} \ (f \ 0)) = f \ N \]

\[ \text{tail} \ (\text{tail} \ (f \ (S \ n))) = f \ n \]
Results of paper in POPL (2013)

- Development of a recursive simply typed calculus (no termination check).
- Allows to derive schemata for pattern/copattern matching.
- Proof that subject reduction holds.

\[ t : A, \quad t \rightarrow t' \text{ implies } t' : A \]

- Subject reduction fails when using codata types in combination with the equality type (e.g. in Coq and early versions of Agda).
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Unnesting of Pattern/Copattern Matching
Consider Example from above

\[ f : \mathbb{N} \rightarrow \text{Stream} \]

\[
\begin{align*}
\text{head} \ (f \ n) &= n \\
\text{head} \ (\text{tail} \ (f \ n)) &= n \\
\text{tail} \ (\text{tail} \ (f \ 0)) &= f \ N \\
\text{tail} \ (\text{tail} \ (f \ (S \ n))) &= f \ n
\end{align*}
\]

We show how this example can be reduced to unnested (co)pattern matching.

In a second step (not shown today) one can reduce it to primitive (co)recursion operators.
Unnesting of Nested (Co)Pattern Matching

We follow the steps in the pattern matching:
We start with

\[
f : \mathbb{N} \rightarrow \text{Stream}
\]

\[
\text{head } (f \ n) = n
\]

\[
\text{tail } (f \ n) = ?
\]
**Copattern matching** on tail \((f \ n)\):

\[
\begin{align*}
    f : \mathbb{N} & \to \text{Stream} \\
    \text{head} \ (f \ n) & = \ n \\
    \text{head} \ (\text{tail} \ (f \ n)) & = \ n \\
    \text{tail} \ (\text{tail} \ (f \ n)) & = \ ? \\
\end{align*}
\]

corresponds to

\[
\begin{align*}
    f : \mathbb{N} & \to \text{Stream} \\
    \text{head} \ (f \ n) & = \ n \\
    \text{tail} \ (f \ n) & = \ g \ n \\
\end{align*}
\]

\[
\begin{align*}
    g : \mathbb{N} & \to \text{Stream} \\
    \text{head} \ (\text{tail} \ (f \ n)) & =) \ \text{head} \ (g \ n) = \ n \\
    \text{tail} \ (\text{tail} \ (f \ n)) & =) \ \text{tail} \ (g \ n) = \ ? \\
\end{align*}
\]
**Pattern matching** on \( \text{tail} \left( \text{tail} \left( f \ n \right) \right) \):

\[
\begin{align*}
  f : \mathbb{N} \rightarrow \text{Stream} \\
  \text{head} \left( f \ n \right) &= n \\
  \text{head} \left( \text{tail} \left( f \ n \right) \right) &= n \\
  \text{tail} \left( \text{tail} \left( f \ 0 \right) \right) &= f \ N \\
  \text{tail} \left( \text{tail} \left( f \left( S \ n \right) \right) \right) &= f \ n
\end{align*}
\]

corresponds to

\[
\begin{align*}
  f : \mathbb{N} \rightarrow \text{Stream} \\
  \text{head} \left( f \ n \right) &= n \\
  \text{tail} \left( f \ n \right) &= g \ n
\end{align*}
\]

\[
\begin{align*}
  g : \mathbb{N} \rightarrow \text{Stream} \\
  \text{head} \left( \text{tail} \left( f \ n \right) \right) &=) \text{head} \left( g \ n \right) = n \\
  \text{tail} \left( \text{tail} \left( f \ n \right) \right) &=) \text{tail} \left( g \ n \right) = k \ n
\end{align*}
\]

\[
\begin{align*}
  k : \mathbb{N} \rightarrow \text{Stream} \\
  \text{tail} \left( \text{tail} \left( f \ 0 \right) \right) &=) k \ 0 = f \ N \\
  \text{tail} \left( \text{tail} \left( f \left( S \ n \right) \right) \right) &=) k \ \left( S \ n \right) = f \ n
\end{align*}
\]
Conclusion

- Principle of induction is well established and makes proofs much easier.
- In theoretical computer science coinductive principles occur frequently.
  - Main reason: interactive programs running continuously in various frameworks (imperative, object-oriented, process-calculi)
- Coalgebras as being defined by their eliminators rather than infinite applications of constructors makes clear when recursive calls are allowed.
- Proofs by coinduction in the above situation can be carried out similarly as proofs by induction.
- Main difficulty: when are we allowed to apply co-IH?
  - In the corecursion step we have a proof obligation, and can use the co-IH to prove it.
Conclusion

- Copattern matching as the dual of pattern matching.
  - Pattern matching is an elimination principle for inductive types (initial algebras).
  - Copattern matching is an introduction principle for coinductive types (final coalgebras).
- Mixed pattern and copattern matching can be reduced to simple pattern and copattern matching.