Proof Theory of Martin-Löf Type Theory

Anton Setzer

Swansea University, Swansea UK

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From Hilbert to Gentzen

- Hilbert's program.
  - Proof consistency of mathematical theories by finitary methods.
- Doesn't work because of Gödel's Incompleteness theorem.
- Gentzen: Reduction of consistency of PA to well-foundedness of ordinal notation systems.

Gentzen's Argument (Modern Version)

- Essentially
  - interpret proofs of PA as infinitary proofs in a semi formal system, e.g. induction interpreted as

$$\dfrac{A(0) \qquad \dfrac{A(0) \qquad A(0)\rightarrow A(1)}{A(1)} \qquad \dfrac{\dfrac{A(0) \qquad A(0)\rightarrow A(1)}{A(1)} \qquad A(1)\rightarrow A(2)}{A(2)} \qquad \cdots}{\forall x.A(x)}$$

Gentzen's Argument (Modern Version)

- If you omit the cut rule

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}\ (\mathrm{Cut})$$

  the calculus is just a truth definition, which can only prove true formulas.
- Proof that cuts can be eliminated using induciton over the height of trees.
- Height can be measured by ordinals, therefore induction over trees can be replaced by transfinite induction over an ordinal notation system of strength ϵ_0.
- So

$$\mathrm{PRA} + \mathrm{TI}(\epsilon_0) \vdash \mathrm{Cons}(PA)$$

Wellfoundedness of Ordinal Notation System

- In case of ϵ_0 some insight into the well-foundedness of the ordinal notation system can be achieved.
- Works as well for theories up to strength $(\Pi^1_1 - \mathrm{CA})_0$.
  - Sufficient by reverse mathematics for proving most mathematical theorems.
  - Argument carried out in articles on "Ordinal systems" by the author.
- Beyond $(\Pi^1_1 - \mathrm{CA})_0$ it becomes difficult to get a direct insight into the well-foundedness of the ordinal notation system.

Wellfoundedness of the Ordinal Notation System

- Need for a second theory in which we can prove the well-foundedness of the ordinal notation system.
  - That theory will then (if it contains PRA) prove the consistency of the theory in question.
    - In fact usually more follows, at least validity of Π^0_2-sentences.
  - Requires a theory for which we have an insight that everything proved in it is valid.
  - Most successful (but not necessarily only) approach: constructive theories.
  - Candidates could be
    - Frege structures,
    - Feferman's systems of explicit mathematics
    - Martin-Löf Type Theory.

Wellfoundedness of the Ordinal Notation System

- Most effort has been taken to develop Martin-Löf Type Theory for that purpose.
  - Argument for validity of its judgements: meaning explanations.
    - If argument formalised mathematically we prove the consistency of the theory in question.
    - Requires by Gödel's Second Incompleteness theorem more strength than the theory itself.
    - Therefore consistency argument needs to be necessarily philosophical in nature.
    - Each consistency argument neeeds to rely on a philosophical argument (even if it is not reflected properly).
  - According to Martin-Löf his type theory is the most serious attempt to create a theory where we have an insight into the validity of its judgements.

Two Step Approach

So we arrive at a two step approach:

- First step:
  - Prove the consistency of a theory T in PRA extended by transfinite induction up to over an ordinal notation system OT.
- Second Step:
  - Proof well-foundedness (transfinite induction) of OT in an extension of Martin-Löf's Type Theory ML^+.
- Since PRA can be embedded into ML^+, we obtain that

$$\mathrm{ML}^+ \vdash \mathrm{Cons}(T)$$

- We obtain even usually more namely that all Π^0_2-statements of T are provable in ML^+.

Need for Proof Theoretically Extensions of Martin-Löf Type Theory

- Needed: development of strong extensions of Martin-Löf Type Theory with an insight into the validity of what can be shown into it.
- Applications:
  - Discovery of advanced data structures for use in programming.

Some Type Theoretic Notations

- We have judgements:

$$a : A \qquad\qquad A : \mathrm{Set}$$

- The latter expresses that A is a small type (= Set)
- We have the dependent function type:

$$(x : A) \to B$$

  - Elements are functions f mapping $a : A$ to $f\ a : B[x := a]$.
  - Example Matrix multiplication:

$$\mathrm{matmult} : (n, m, k : \mathbb{N}) \to \mathbb{R}^{n,m} \to \mathbb{R}^{m,k} \to \mathbb{R}^{n,k}$$

Version of MLTT of Strength PA

- MLTT (which includes $\mathbb{N}$) and a small universe we obtain the strength of PA (ϵ_0).
  - We call this microscopic universe Atom and the theory $\mathrm{MLTT} + \mathrm{Atom}$.
  - Only problematic principle: $\mathbb{N}$.
  - Trust in the validity requires an insight into the understanding the concept of a least set introduced by finitary introduction rules.
  - Insight closely related to the concept of time.

Principles for Adding Strength

- Two principles added in order to increase the strength of type theory:
  - The W-type.
  - Universes.

The W-Type

- **Formation rule:**

$$\frac{A : \mathrm{Set} \qquad B : A \to \mathrm{Set}}{\mathrm{W}(A, B) : \mathrm{Set}}$$

- **Introduction rule:**

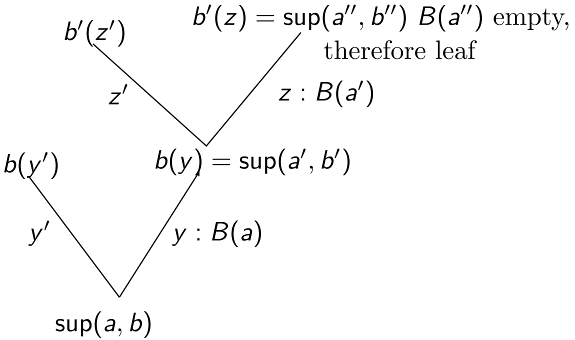
$$\frac{a : A \qquad b : B(a) \to \mathrm{W}(A, B)}{\mathsf{sup}(a, b) : \mathrm{W}(A, B)}$$

- **Elimination/equality rule:**

  Induction over trees.

W-Type

$\mathrm{W}(A, B)$ is the type of well-founded recursive trees with branching degrees $(B(a))_{a:A}$.



Understanding of the W-Type

- Understanding the meaning of the W-type requires understanding the existence of a type containing exactly those elements introduced by the introduction rules.
- Leastness more difficult than elements of the W-type not introduced in finitely many steps.
  - We can only draw an analogy from the concept of time.

Strength of the W-Type

- $\mathrm{MLTT} + \mathrm{Atom} + \mathrm{W}$ has strength of finitely iterated inductive definitions or $(\Pi^1_1 - \mathrm{CA})_0$.
  - In proof theory considered as truely impredicative theory.
  - Requires substantially more complex techniques and ordinal notation systems in order to understand them.
    - As mentioned before some direct insight into the well-foundedness can still be obtained.
- By the above mendioned results of reverse mathematics sufficient for proving most mathematical theorems.
- So most mathematics actually used is secured, if we trust in validity of judgements in $\mathrm{MLTT} + \mathrm{Atom} + \mathrm{W}$.

Universes

- A universe is a family of sets
- Given by
  - a set $\mathrm{U} : \mathrm{Set}$ of **codes** for sets,
  - a **decoding function** $\mathrm{T} : \mathrm{U} \to \mathrm{Set}$.

Universes

- **Formation rules:**

$$\mathrm{U} : \mathrm{Set} \qquad \mathrm{T} : \mathrm{U} \to \mathrm{Set}$$

- **Introduction and Equality rules:**

$$\widehat{\mathbb{N}} : \mathrm{U} \qquad \mathrm{T}(\widehat{\mathbb{N}}) = \mathbb{N}$$

$$\frac{a : \mathrm{U} \qquad b : \mathrm{T}(a) \to \mathrm{U}}{\widehat{\Sigma}(a, b) : \mathrm{U}}$$

$$\mathrm{T}(\widehat{\Sigma}(a, b)) = \Sigma(\mathrm{T}(a), \mathrm{T} \circ b)$$

  Similarly for other type formers (except for U).

- **Elimination/equality rules:** Induction over U.
  (Not needed in order to obtain their strength).

Understanding of Universes

- Insight into validity of rules for universes is not much more complex than insight into validity of rules for W-type.
- $\mathrm{MLTT} + \mathrm{W} + \mathrm{U}$ has strength KPI^{+} (AS) which is a slight extension of KPI which is Kripke Platek set theory plus one recursively inaccessible ordinal.
  - KPI corresponds roughly speaking to an inductive definition which refers to closure under another inductive definition.
- Proof theoretically there are some complications but not dramatic ones.

Understanding of Universes

- In the ordinal notation system, something happens when making the step from $(\Pi^1_1 - \mathrm{CA})_0$ to $(\Pi^1_1 - \mathrm{CA})$.
  - Strength of KPI is $\Delta^1_2 - \mathrm{CA} + \mathrm{BI}$ which is much stronger than $(\Pi^1_1 - \mathrm{CA})$.

Generalisation to Inductive-Recursive Definitions

- Inductive-Recursive Definitions originally defined by Dybjer, closed formalisation by Dybjer + AS.
- Definition of a type theory containing all standard (inductive) definitions, universes, and many generalisations.
- Generalise the principles.

Induction-Recursion

- We have one set $\mathrm{U} : \mathrm{Set}$ with constructors:

$$\mathrm{C} : \underbrace{(a : A)}_{\text{non-inductive argument}}$$

$$\rightarrow \quad \underbrace{(b : B\ a \rightarrow \mathrm{U})}_{\text{inductive argument depending on } a}$$

$$\rightarrow \quad \underbrace{(c : (x : D\ a) \times \mathrm{T}\ (b\ (f\ x)))}_{\text{non-inductive arguments depending on } a \text{ and } \mathrm{T} \circ b}$$

$$\rightarrow \quad \cdots$$

$$\rightarrow \quad \mathrm{U}$$

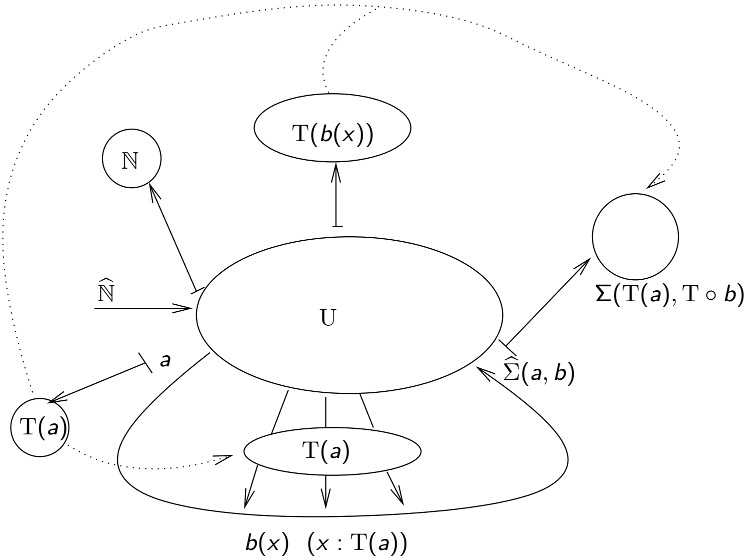
Induction-Recursion

- We have $\mathrm{T} : \mathrm{U} \to \mathrm{Set}$ with recursive equations for each constructor:

$$\mathrm{T}\ (\mathrm{C}\ a\ b\ c\ \cdots) = t[a, \mathrm{T} \circ b, c, \ldots] : \mathrm{Set}$$

- Generalisation to $\mathrm{T}\ u : D$ for some type D.
- If $D = \{*\}$, we obtain the special case of inductive definitions.

Universe


T(b(x))
ℕ
$\widehat{\mathbb{N}}$
U
$\Sigma(\mathrm{T}(a), \mathrm{T} \circ b)$
a
$\widehat{\Sigma}(a, b)$
T(a)
T(a)
$b(x) \quad (x : \mathrm{T}(a))$

Strength of Induction-Recursion

- Proof theoretic strength in $[\mathrm{KPM}, \mathrm{KPM}^+]$.
- $\mathrm{KPM} =$ Kripke-Platek set theory plus one recursively Mahlo ordinal.
- $\mathrm{KPM}^+ =$ slight extension of KPM.
- (Upper bound not formally proved yet).

Steps Towards Mahlo

- First step beyond standard universe
  - The super universe (Palmgren).
  - He introduced a universe V,
  - together with a universe operator $\mathrm{U} : \mathrm{Fam}(\mathrm{V}) \to \mathrm{V}$,
    - $\mathrm{Fam}(\mathrm{V})$ is the set of families of sets in V indexed over elements of V, roughly speaking

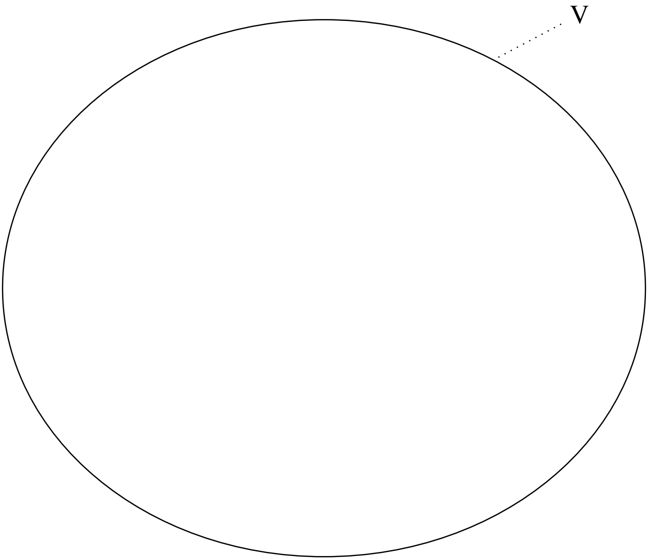
$$\{(B_x)_{x:B} | B : \mathrm{V}, \quad x : B \Rightarrow B_x : \mathrm{V}\}$$

  - s.t. for any family of sets A in V, $\mathrm{U}(A)$ is a universe containing all elements of A.

Steps Towards Mahlo

- A Universe is a family of sets closed under constructions for forming sets.
- We can now form a universe, closed under the formation of the next universe above a family of sets.
- (The next slide doesn't exhaust the strength, it shows only universes containing one set, not universes containing family of sets)

Illustration of the Super Universe



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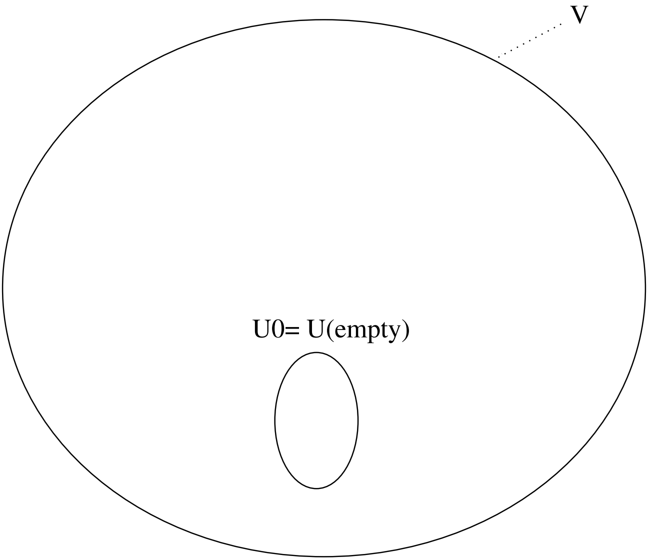

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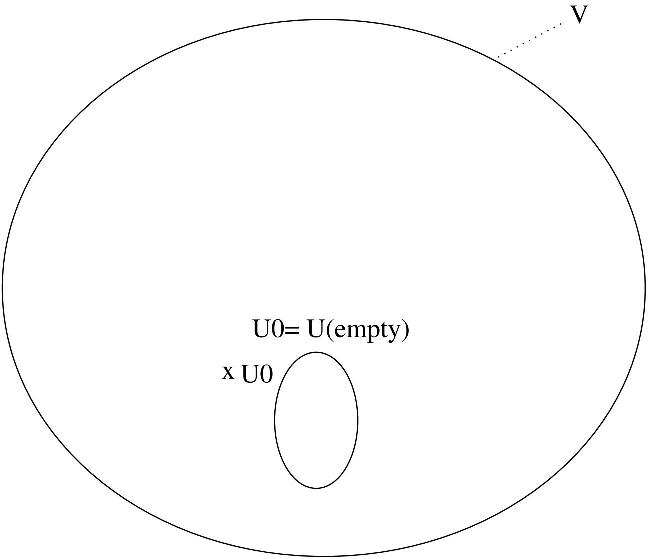

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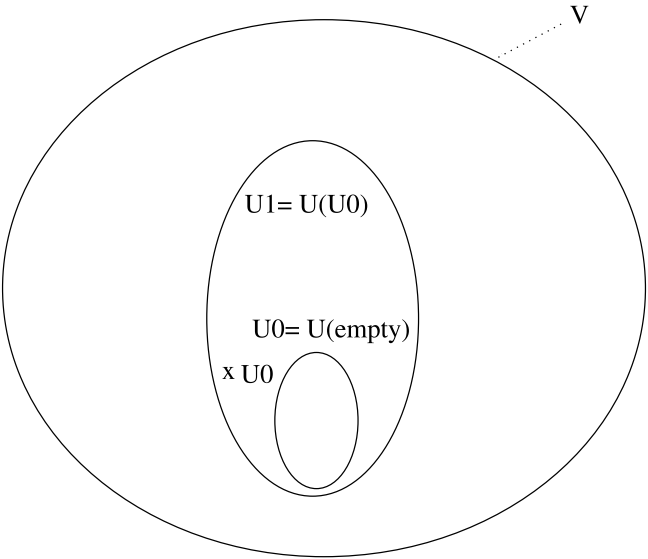

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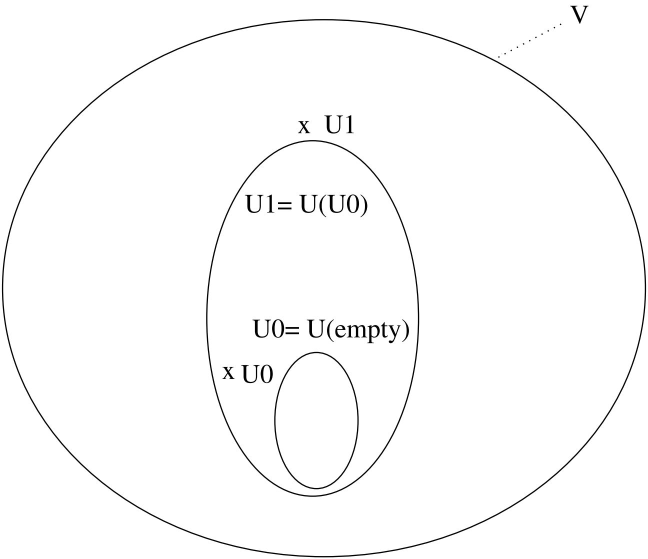

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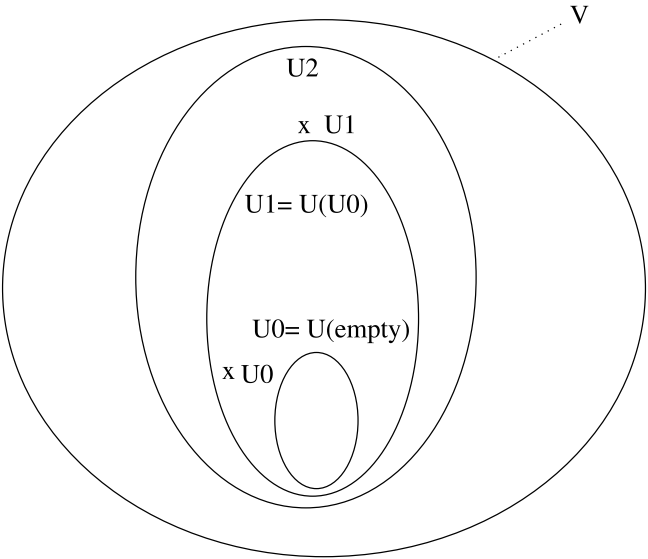

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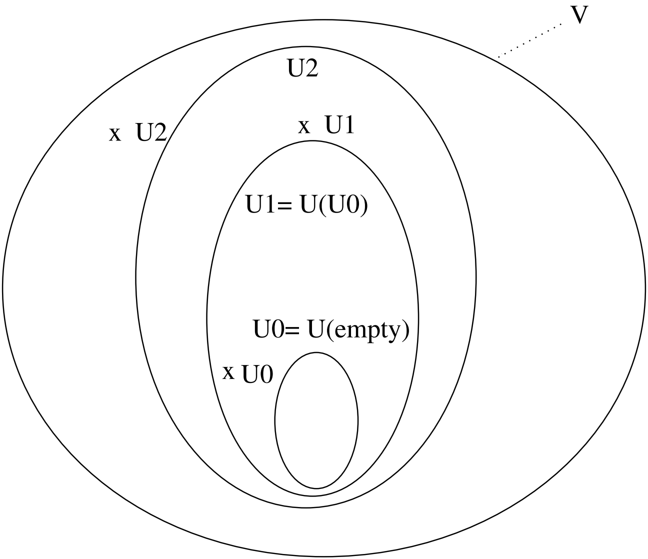


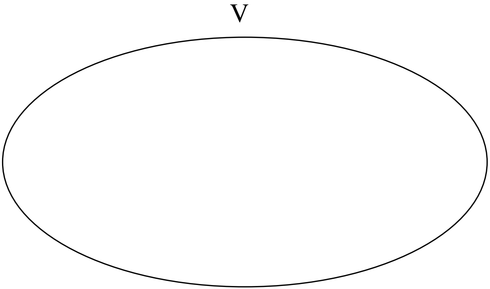
Super^n-Universes

- The above can be continued: We can form a
  - super^2-universe V,
  - closed under a super-universe operator, forming a super universe above a family of sets in V.
- And we can iterate the above n-many times, and even go beyond.
- Up to now everything was inductive-recursive

Mahlo Universe

- The Mahlo universe is
  - a universe $\underset{\sim}{\mathrm{V}}$,
  - which has not only subuniverses corresponding to some operators, but subuniverses corresponding to all operators it is closed under:
  - for every universe operator on V,
    - i.e. every $f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$,
  - there exists a universe $\underset{\sim}{\mathrm{U}_f}$ closed under f.
  - which is represented in V.

Illustration of the Mahlo Universe



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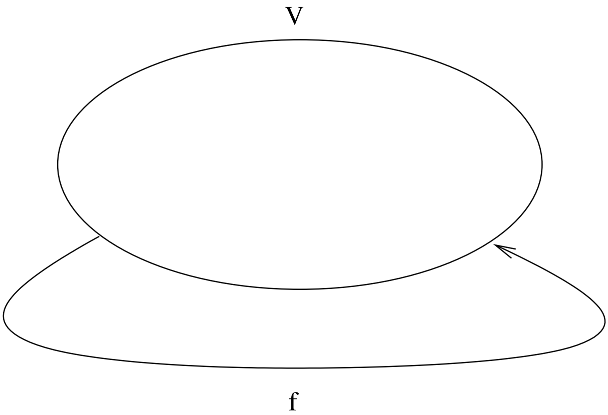

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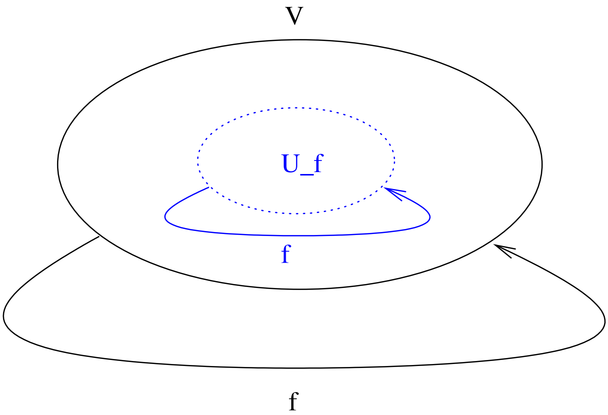

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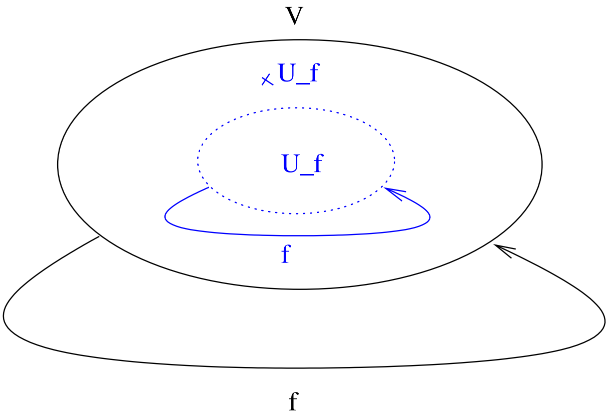


Impredicativity of the Mahlo Universe

- The introduction rule for V introduces for every

$$f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$$

  an element

$$\widehat{\mathrm{U}}_f : \mathrm{V}$$

- Depends on the totality of V.

Red Mahlo Universe

- If we only have Set as a Mahlo universe, then we obtain a construction which is an example of inductive-recursive definitions.
- Strenght of IRD primarily based on the fact that we can define a universe U_f closed under a function

$$f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$$

- So we have

$$f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set}) \Rightarrow \mathrm{U}_f : \mathrm{Set}$$

Red Mahlo Universe

- Therefore Set is essentially a Mahlo universe, called "the red Mahlo universe".
- Highly depends on the use of the logical framework.
- Strength of the Red Mahlo universe is expected to be KPM.

Red Mahlo Universe

- The well-ordering proof requires as well that we have large elimination for W-type and $\mathbb{N}$ (elimination into an arbitrary type rather than a set).
  - Can be omitted by forming suitable indexed higher type universes which are sufficiently closed.
- The red Mahlo universe is usually accepted, whereas the black Mahlo universe is often rejected.
- However, the red Mahlo universe is essentially as impredicative as the black Mahlo universe.
- It is accepted because one considers Set as an "open concept".
- However introduction rule for Set depends on the the set of total functions

$$f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$$

Problems of Mahlo Universe

- This section is joint work with R. Kahle.
- Elements of V are constructed, depending on total functions

$$f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$$

- Therefore V has an impredicative definition.
- However, for defining U_f, only the restriction of f to $\mathrm{Fam}(\mathrm{U}_f)$ is required to be total.
- In order to define functions for which this restrictions is total, we need to define candidates of U_f for arbitrary (not necessarily total) f.
- Problem: In Martin-Löf Type Theory all functions are total.
- In Feferman's explicit mathematics reference to arbitrary terms possible.

Extended Predicative Mahlo (in Explicit. Mathematics)

- We will use syntax borrowed from type theory in Explicit Mathematics.
  - but $a \in B$ instead of $a : B$.
- Explicit mathematics more Russell-style, therefore we can have $\mathrm{V} \in \mathrm{Set}$, $\mathrm{V} \subset \mathrm{Set}$.
- We can encode $\mathrm{Fam}(\mathrm{V})$ into V, therefore need only to consider functions $f : \mathrm{V} \to \mathrm{V}$.

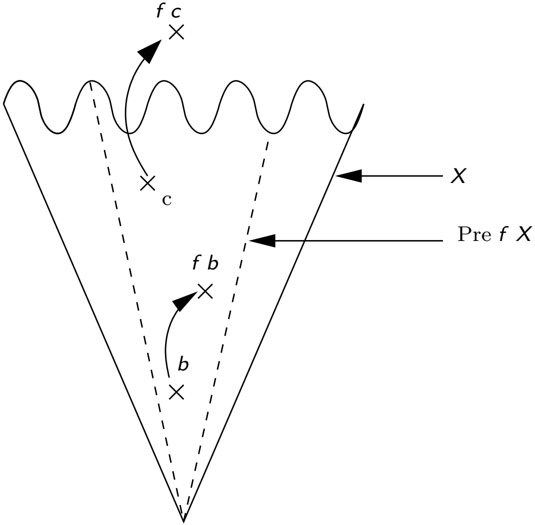
Step 1: Define $\mathrm{Pre}\ f\ X$

- Define V to be closed under universe constructions for explicit mathematics.
- Define for $f, X \in \mathrm{Set}$, $X \subseteq \mathrm{Set}$

$$\mathrm{Pre}\ f\ X \in \mathrm{Set} \qquad \mathrm{Pre}\ f\ X \subseteq X$$

- $\mathrm{Pre}\ f\ X$ is the least subset of X closed under universe constructions and f relative to X.
- So, if
  - result of applying a universe operator to $\mathrm{Pre}\ f\ X$ is in X, then add it to $\mathrm{Pre}\ f\ X$.
  - result of applying f to an elemnet of $\mathrm{Pre}\ f\ X$ is in X, then add it to X.

Pre f X

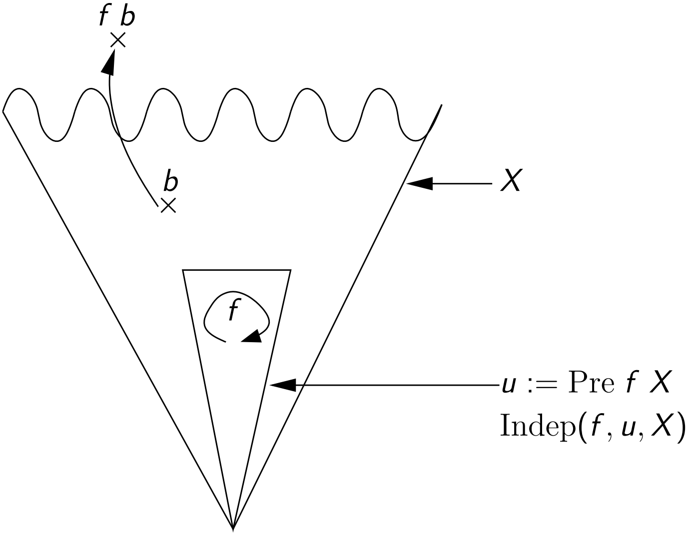


Step 2: Independence of Pre f X

- If, whenever a universe construction or f is applied to elements of $\mathrm{Pre}\ f\ X$ we get elements in X, then $\mathrm{Pre}\ f\ X$ is independent of future extensions of X.

$$\begin{aligned}\mathrm{Indep}(f, \mathrm{Pre}\ f\ X, X) := {} & (\forall a \in \mathrm{Pre}\ f\ X.\ \forall b \in a \to \mathrm{Pre}\ f\ X.\ \mathrm{j}\ a\ b \in X) \\ & \wedge\ \cdots \\ & \wedge\ \forall a \in \mathrm{Pre}\ f\ X.\ f\ a \in X\end{aligned}$$

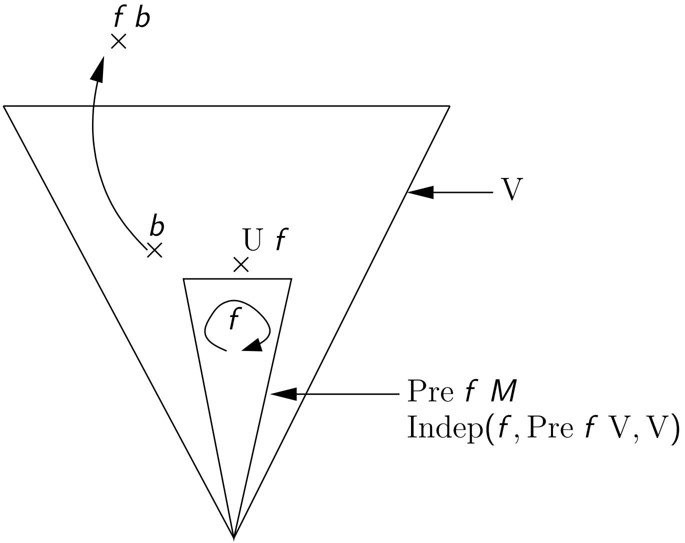
Indpt



Step 3: Introduction Rule for V

- $$\forall f.\,\mathrm{Indep}(f, \mathrm{Pre}\ f\ \mathrm{V}, \mathrm{V}) \to (\mathrm{U}\ f \in \mathrm{Set} \wedge \mathrm{U}\ f =_{\mathrm{ext}} \mathrm{Pre}\ f\ \mathrm{V} \wedge \mathrm{U}\ f \in \mathrm{V}) \quad .$$

Introduction Rule for V



Interpretation of Axiomatic Mahlo

- One can show:

$$\forall f \in \mathrm{V} \to \mathrm{V}.\ \mathrm{Indep}(f, \mathrm{Pre}\ f\ \mathrm{V}, \mathrm{V})$$

therefore

$$\forall f \in \mathrm{V} \to \mathrm{V}.\ \mathrm{U}\ f \in \mathrm{V}\ \wedge\ \mathrm{Univ}(f)\ \wedge\ f \in \mathrm{U}\ f \to \mathrm{U}\ f$$

- So V closed under axiomatic Mahlo constructions.
- Therefore extended predicative Mahlo has at least strength of axiomatic Mahlo.

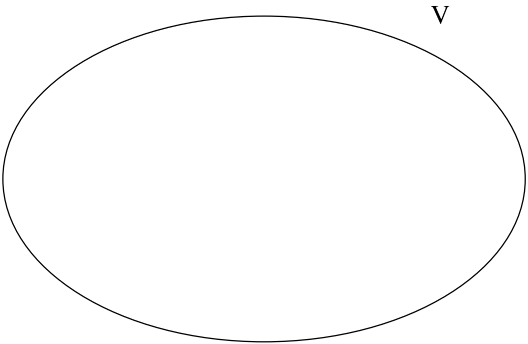
4. Π_3-Reflection

- First step: Formation of Hyper-Mahlo Universes:
  - A hyper Mahlo universe is a unverse V, T s.t. for every

$$f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V})$$

  there exists a subuniverse of V
    - closed under f,
    - which is Mahlo.

Illustration of the Hyper Mahlo Univ.



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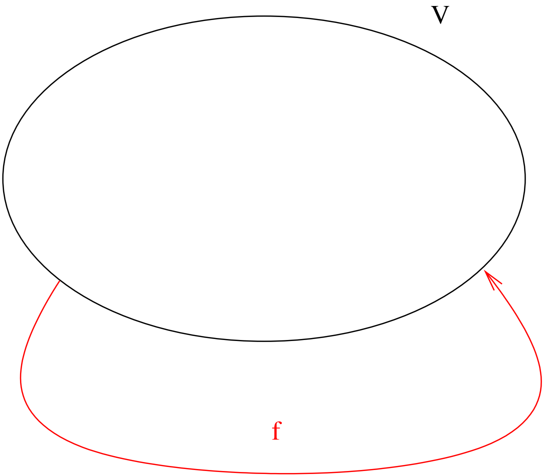

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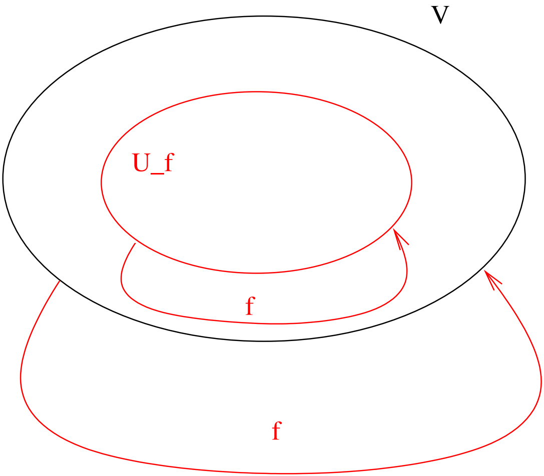

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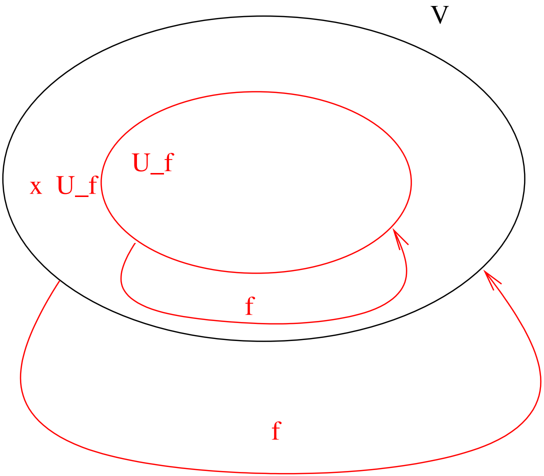

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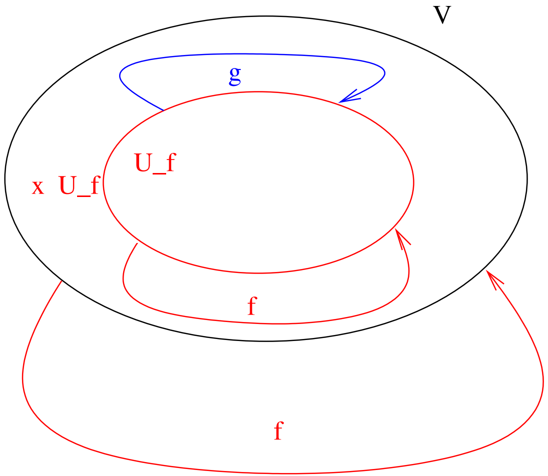

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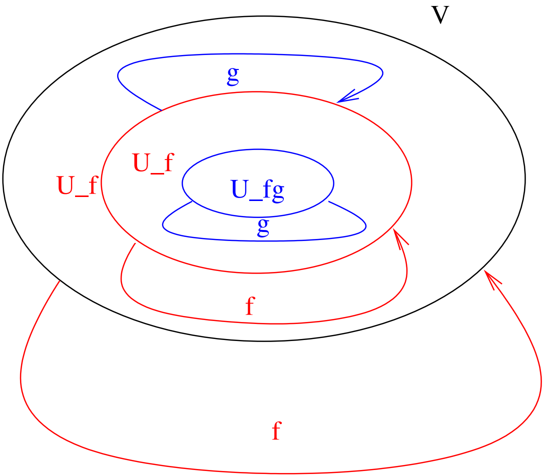

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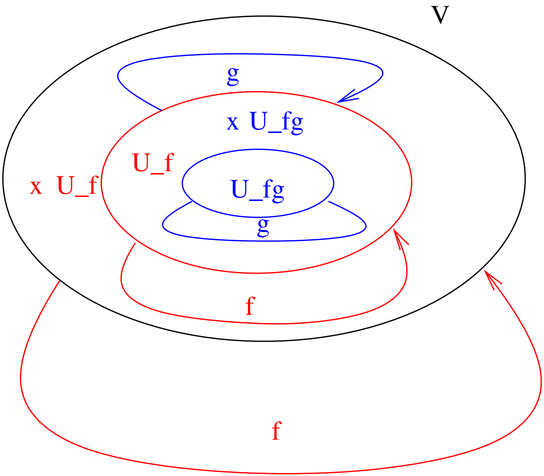

Extensions

- Hyper^n-Mahlo Universe: Straightforward Extension.
- Autonomous Mahlo Universe consists of
  - A universe V, S.
  - The set of Mahlo degrees M which is essentially $\mathrm{W}x : \mathrm{V}.\mathrm{S}(x)$.
  - Rules expressing that V is Hyper^w-Mahlo for any $w : \mathrm{M}$.

Π_3-Reflecting Universe

- In case of the Π_3-reflecting universe, we obtain Hyper^d-Mahlo universes for Mahlo degrees d.
  - Mahloness of a hyper^d-Mahlo universe depends locally on the hyper^d-Mahlo universe as well.
- Mahlo degrees are introduced by an introduction rule similar to that of the Mahlo universe:
  - For every

    $$f : \mathrm{Fam}(\mathrm{V}) \to \mathrm{Fam}(\mathrm{V}, \mathrm{MDegree})$$

    we define a new

    $$\mathrm{degree}(f) : \mathrm{MDegree}$$

Π_3-Reflecting Universe

- If Mahlo universe is acceptable, step towards to the Π_3-reflecting universe is essentially a technical difficulty.

Conclusion

- Consistency problem leads naturally to proofs of consistency using transfinite induction.
- Because of lack of direct insight into well-foundedness need to prove well-foundedness of ordinal notation system in constructive theories.
- Up to inductive-reducrsive definition analysis not controversal.
- Mahlo universe is controversal because of impredicative characater.
- Solution by giving an extended predicative Mahlo universe.
- Extension to Π_3-reflecting universe.