A model for the extended predicative Mahlo Universe

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Explicit Mathematics

Extended Predicative Mahlo

Model for Extended Predicative Mahlo Universe

Explicit Mathematics

- Explicit mathematics based on term language where terms can denote elements of sets and sets.
- No restriction on application, succ(nat) is a term.
- $a \mathrel{\dot{\in}} b$ for a is an element of the set denoted by b.
- $\Re(a)$ for a is a name, i.e. denotes a set.

Inductive Generation

- $\mathsf{i}(u, v)$ denotes the accessible part of the relation v on domain u.

- $\mathcal{C}losed^{\mathsf{i}}(a, b, S) := \forall x \dot{\in} a.(\forall y \dot{\in} a.(y, x) \dot{\in} b \rightarrow y \in S) \rightarrow x \in S$
- $\Re(a) \wedge \Re(b) \rightarrow \exists X.\Re(\mathsf{i}\,(a, b), X) \wedge \mathcal{C}losed^{\mathsf{i}}(a, b, X)$
- $\Re(a) \wedge \Re(b) \wedge \mathcal{C}losed^{\mathsf{i}}(a, b, \phi) \rightarrow \forall x \dot{\in} \mathsf{i}\,(a, b).\phi(x)$

Universes

- $\mathcal{C}los^{\text{univ}}(W, x)$ expresses that x is formed using the above universe operations (excluding i) from elements in W.
- $\mathcal{U}niv(W) := (\forall x \in W.\Re(x)) \land \forall x.\mathcal{C}los^{\text{univ}}(W, x) \to x \in W$.
- $\text{Univ}(t) := \exists X.\Re(t, X) \land \mathcal{U}niv(X)$.

Axiomatic Mahlo Universe

- To deal with size problem, when working in type theory and explicit mathematics one needs large universes.
- Allows as well to obtain proof theoretic stronger theories.
- Axiomatic Mahlo universe is a universe M such as for every $a \in \mathrm{M}$ and $f \in \mathrm{M} \to \mathrm{M}$ there exists a subuniverse $\mathrm{u}(a, f)$ of M which is closed under a and f and an element of M.

Illustration of the Mahlo Universe

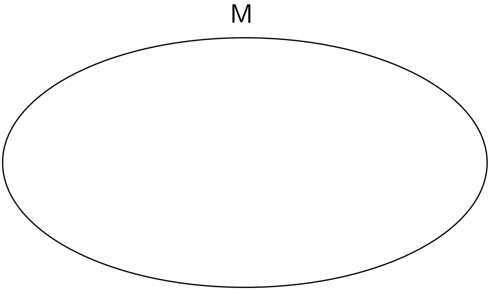
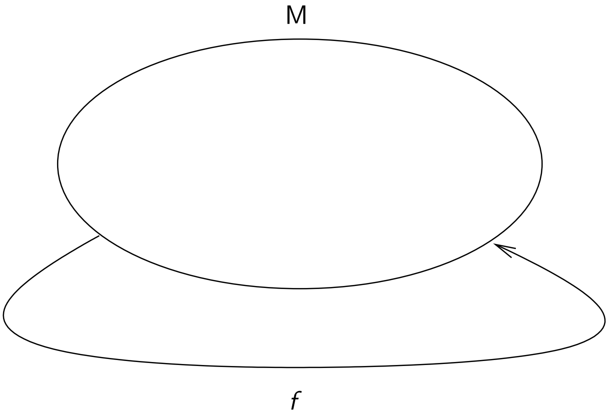


Illustration of the Mahlo Universe



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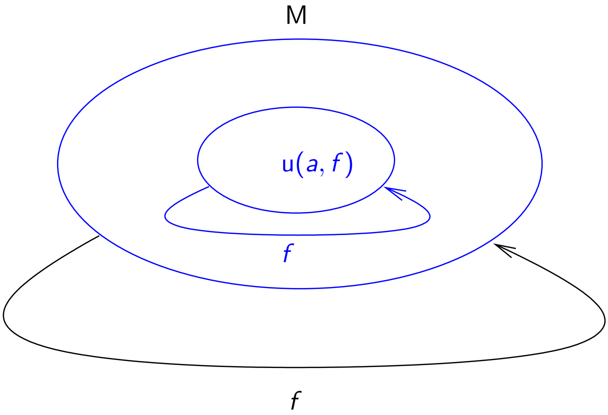

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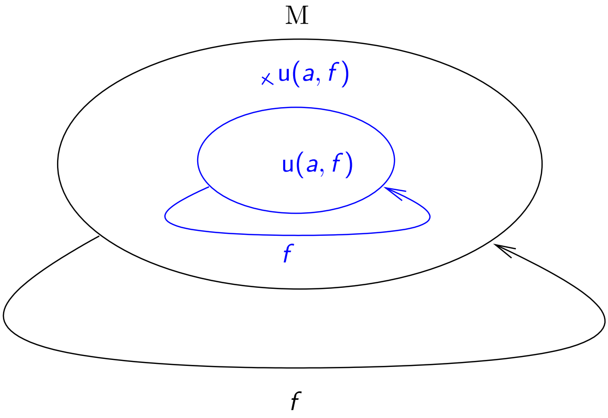


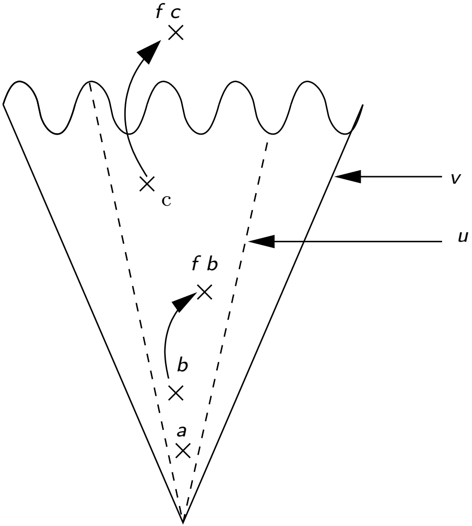
From Axiomatic to Extended Predicative Mahlo

- Problem: introduction rule for $\mathrm{u}(a, f)$ depends on total functions $f : \mathrm{M} \to \mathrm{M}$, which is impredicative
- Totality on M is not really needed, only that f is total on u.
- Extended predicative Mahlo universe formalises this.

Pre-Universe

- Formula expressing that v is a relative preuniverse:

$$
\begin{aligned}
\mathrm{RPU}(a, f, v, u) := \quad & (\forall x.\mathcal{C}los^{\mathrm{univ}}(u, x) \wedge x \mathrel{\dot{\in}} v \rightarrow x \mathrel{\dot{\in}} u) \wedge \\
& (a \mathrel{\dot{\in}} v \rightarrow a \mathrel{\dot{\in}} u) \wedge \\
& (\forall x \mathrel{\dot{\in}} u.f\, x \mathrel{\dot{\in}} v \rightarrow f\, x \mathrel{\dot{\in}} u)
\end{aligned}
$$

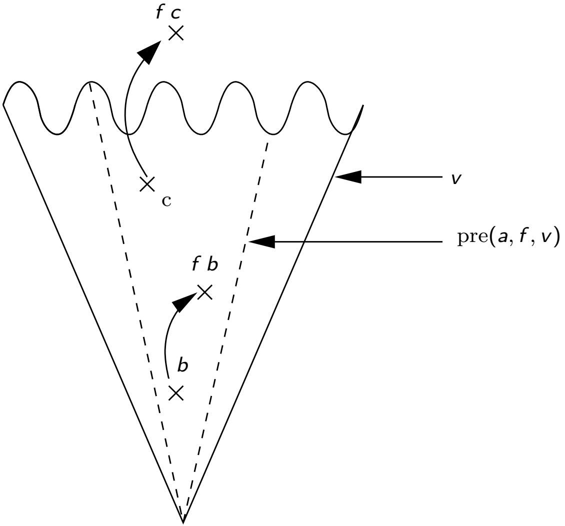

f c
c
v
u
f b
b
a

Least pre-universes

$$\Re_{\Re}(v) \to \mathsf{RPU}(a, f, v, \mathrm{pre}\,(a, f, v)).$$

$$\Re_{\Re}(v) \land \mathsf{RPU}(a, f, v, \phi) \to \forall x \mathrel{\dot{\in}} \mathrm{pre}\,(a, f, v).\phi(x)$$

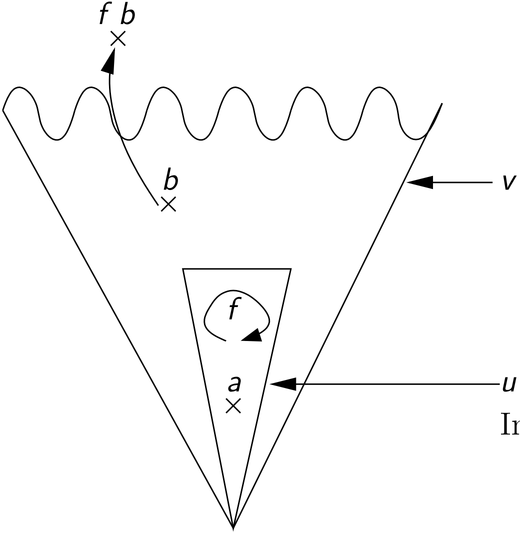
pre(a,f,v)



Independence of pre(a,f,v)

$$
\begin{aligned}
\mathrm{Indep}(a, f, v, u) := \quad & (\forall x.\mathcal{C}los^{\text{`univ}}(u, x) \to x \in v) \land \\
& a \mathrel{\dot{\in}} v \land \\
& (\forall x \mathrel{\dot{\in}} u.f\, x \mathrel{\dot{\in}} v)
\end{aligned}
$$

Indep(a,f,v,u)



$u := \mathrm{pre}(a, f, v)$
$\mathrm{Indep}(a, f, v, u)$

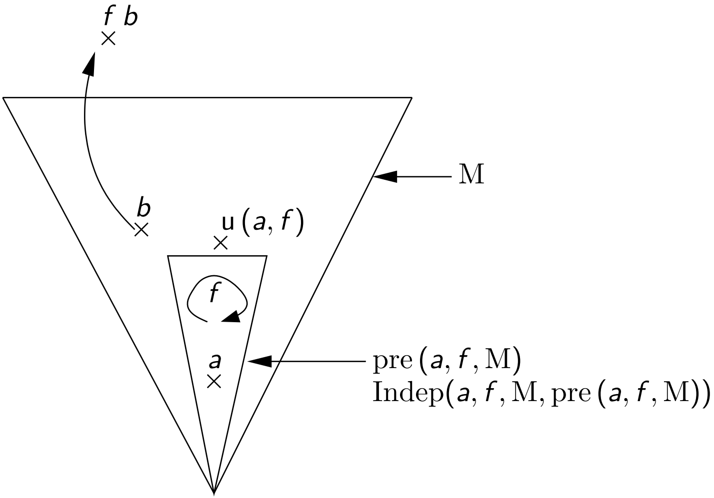
Axioms for M

$$\mathsf{Univ}(\mathrm{M}) \wedge \mathsf{i} \in (\forall a, b \,\dot{\in}\, \mathrm{M} \rightarrow \mathsf{i}(a, b) \,\dot{\in}\, \mathrm{M})$$

$$\mathrm{Indep}(a, f, \mathrm{M}, \mathrm{pre}\,(a, f, \mathrm{M})) \rightarrow \mathsf{u}\,(a, f) \,\dot{\in}\, \mathrm{M} \wedge \mathsf{u}\,(a, f) \doteq \mathrm{pre}\,(a, f, \mathrm{M})$$

Induction expressing M is least set with these closure properties can be added as well.

Introduction Rule for M



Explicit Mathematics

Extended Predicative Mahlo

Model for Extended Predicative Mahlo Universe

Model given by a Relation

- Define codes for terms such as

$$\widehat{\mathrm{int}}(a, b) := \langle 3, a, b \rangle$$

- Let predicates $P \subseteq \mathbb{N}^3$ encode relations $\Re_P, \in_P, \notin_P$ by

$$\begin{array}{lcl} \Re_P(a) & := & P(a, 0, 0), \\ b \in_P a & := & P(a, b, 1), \\ b \notin_P a & := & \Re_P(a) \wedge \neg(b \in_P a) \end{array}$$

Operator for Universe Constructions

$$\Re_P^{\mathrm{int}}(a, u, v) \quad := \quad a = \widehat{\mathrm{int}}(u, v) \wedge \Re_P(u) \wedge \Re_P(v)$$

$$\Re_P^{\mathrm{int}}(a) \quad := \quad \exists u, v. \Re_P^{\mathrm{int}}(a, u, v)$$

$$b \in_P^{\mathrm{int}} a \quad := \quad \exists u, v. \Re_P^{\mathrm{int}}(a, u, v) \wedge (b \in_P u \wedge b \in_P v)$$

similarly for other universe constructions

Operator for Universe Constructions

$$\begin{array}{lcl} \mathfrak{R}^{\mathrm{univ}}_P(x) & := & x = \mathsf{nat} \lor x = \mathsf{id} \lor \mathfrak{R}^{\mathrm{int}}_P(x) \lor \cdots \\ a \in^{\mathrm{univ}}_P b & := & a \in^{\mathrm{nat}}_P b \lor a \in^{\mathrm{id}} b \lor a \in^{\mathrm{int}}_P b \lor \cdots \end{array}$$

$$\begin{array}{lcl} \Gamma^{\mathrm{univ}}_P(a, b) & := & b = \mathsf{nat} \lor b = \mathsf{id} \\ & & \lor\ (\exists u, v. b = \widehat{\mathsf{int}}(u, v) \land u \in_P a \land v \in_P a) \\ & & \lor\ \cdots \end{array}$$

Modelling Inductive Generation

$$
\begin{array}{lcl}
\Re_P^{\mathrm{pre-i}}(a,u,v) & := & a = \widehat{\mathrm{i}}(u,v) \wedge \Re_P(u) \wedge \Re_P(v) \\
\Gamma_P^{\mathrm{i,pot}}(a,b) & := & \exists u,v.b = \widehat{\mathrm{i}}(u,v) \wedge u \in_P a \wedge v \in_P a \\
b \in^{\mathrm{i}}_P a & := & \exists u,v.a = \widehat{\mathrm{i}}(u,v) \wedge b \in_P u \\
 & & \quad \wedge \forall x \in_P u.\langle x,b\rangle \in_P v \rightarrow x \in_P a \\
\mathcal{C}losed^{\mathrm{i}}{}_P(u,v) & := & \forall x \in^{\mathrm{i}}_P \widehat{\mathrm{i}}(u,v).x \in_P \widehat{\mathrm{i}}(u,v) \\
\Re^{\mathrm{i}}{}_P(a,u,v) & := & \Re_P^{\mathrm{pre-i}}(a,u,v) \wedge \mathcal{C}losed^{\mathrm{i}}{}_P(u,v) \\
\Re^{\mathrm{i}}{}_P(a) & := & \exists u,v.\Re^{\mathrm{i}}{}_P(a,u,v) \\
\Gamma^{\mathrm{i}}{}_P(a,b) & := & \exists u,v.b = \widehat{\mathrm{i}}(u,v) \wedge \Gamma_P^{\mathrm{i,pot}}(a,b) \wedge \mathcal{C}losed^{\mathrm{i}}{}_P(u,v)
\end{array}
$$

Modelling Pre universes

$$
\begin{array}{lcl}
b \in_P^{\mathsf{pre,pot}} a' & := & \exists a, f, v. a' = \widehat{\mathsf{pre}}(a, f, v) \\
 & & \quad \wedge (b = a \\
 & & \qquad \vee (\exists x \in_P a'. b \simeq \{f\}(x)) \\
 & & \qquad \vee \Gamma_P^{\mathsf{univ}}(\widehat{\mathsf{pre}}(a, f, v), b)) \\
b \in_P^{\mathsf{pre}} a' & := & b \in_P^{\mathsf{pre,pot}} a' \wedge b \in_P v \\
\mathcal{C}losed_P^{\mathsf{pre}}(a, f, v) & := & \forall b \in_P^{\mathsf{pre}} \widehat{\mathsf{pre}}(a, f, v). b \in_P \widehat{\mathsf{pre}}(a, f, v) \\
\mathrm{Indep}^{\mathsf{pre}}(a', v) & := & \exists a, f. a' = \widehat{\mathsf{pre}}(a, f, v) \\
 & & \wedge \forall b \in_P^{\mathsf{pre,pot}} a'. b \in_P v \\
\Re_P^{\mathsf{pre}}(a', a, f, v) & := & a' = \widehat{\mathsf{pre}}(a, f, v) \\
 & & \wedge\, \mathcal{C}losed_P^{\mathsf{pre}}(a, f, v) \\
 & & \wedge (\Re_P(v) \vee \mathrm{Indep}^{\mathsf{pre}}(a', v))
\end{array}
$$

Modelling Pre universes

$$\Re_P^{\mathrm{pre}}(a') \quad := \quad \exists a, f, v. \Re_P^{\mathrm{pre}}(a', a, f, v)$$

Modelling u(a,f)

$$\mathfrak{R}^{\mathsf{u},\mathsf{pot}}_{P}(a',a,f) \quad := \quad a' = \widehat{\mathsf{u}}(a,f) \wedge \mathrm{Indep}^{\mathsf{pre}}(\widehat{\mathsf{pre}}(a,f,\mathsf{M}),\mathsf{M})$$

$$\mathfrak{R}^{\mathsf{u},\mathsf{pot}}_{P}(a') \quad := \quad \exists a,f.\mathfrak{R}^{\mathsf{u},\mathsf{pot}}_{P}(a',a,f)$$

$$\mathfrak{R}^{\mathsf{u}}_{P}(a',a,f) \quad := \quad \mathfrak{R}^{\mathsf{u},\mathsf{pot}}_{P}(a',a,f) \wedge \mathcal{C}losed^{\mathsf{pre}}(\widehat{\mathsf{pre}}(a,f,\mathsf{M}),\mathsf{M})$$

$$\mathfrak{R}^{\mathsf{u}}_{P}(a') \quad := \quad \exists a,f.\mathfrak{R}^{\mathsf{u}}_{P}(a',a,f)$$

$$b \in^{\mathsf{u}}_{P} a' \quad := \quad \exists a,f.\mathfrak{R}^{\mathsf{u}}_{P}(a',a,f) \wedge b \in_{P} \widehat{\mathsf{pre}}(a,f,\mathsf{M})$$

Modelling u(a,f)

$$
\begin{aligned}
b \in_P^{\mathrm{M,pot}} a \quad &:= \quad a = \mathrm{M} \\
&\quad \wedge (\Gamma_P^{\mathrm{univ}}(\mathrm{M}, b) \vee \Gamma_P^{\mathrm{i,pot}}(\mathrm{M}, b) \vee \Re_P^{\mathrm{u,pot}}(b))
\end{aligned}
$$

$$
\begin{aligned}
b \in_P^{\mathrm{M}} a \quad &:= \quad a = \mathrm{M} \\
&\quad \wedge (\Gamma_P^{\mathrm{univ}}(\mathrm{M}, b) \vee \Gamma_P^{\mathrm{i}}(\mathrm{M}, b) \vee \Re_P^{\mathrm{u}}(b))
\end{aligned}
$$

$$
\mathcal{C}losed_P^{\mathrm{M}} \quad := \quad \forall b \in_P^{\mathrm{M,pot}} \mathrm{M}. b \in_P \mathrm{M}
$$

$$
\Re_P^{\mathrm{M}}(a) \quad := \quad a = \mathrm{M} \wedge \mathcal{C}losed_P^{\mathrm{M}}
$$

$\mathcal{A}(P)$

$$\mathcal{A}^{\text{univ}}(P) := \text{Pred}(\Re^{\text{univ}}_P, \in^{\text{univ}}_P)$$

$$\mathcal{A}^{\text{i}}(P) := \text{Pred}(\Re^{\text{i}}_P, \in^{\text{i}}_P)$$

$\ldots$

$$\mathcal{A}(P) := \mathcal{A}^{\text{univ}}(P) \cup \mathcal{A}^{\text{i}}(P) \cup \mathcal{A}^{\text{pre}}(P) \cup \mathcal{A}^{\text{u}}(P) \cup \mathcal{A}^{\text{M}}(P)$$

$$\begin{aligned} \mathcal{A}^0(P) &:= \emptyset \\ \mathcal{A}^{\alpha+1}(P) &:= \mathcal{A}(\mathcal{A}^\alpha(P)) \\ \mathcal{A}^\lambda(P) &:= \bigcup_{\alpha<\lambda} \mathcal{A}^\alpha(P) \end{aligned}$$

$P \preceq Q$

$$
\begin{array}{lll}
P \preceq Q & :\Leftrightarrow & P \subseteq Q \\
 & & \land \forall a, b. \Re_P(b) \to (a \in_P b \leftrightarrow a \in_Q b)
\end{array}
$$

Properties of $\mathcal{A}$

$\emptyset \preceq \mathcal{A}(\emptyset)$

$P \preceq \mathcal{A}(P) \to \mathcal{A}(P) \preceq \mathcal{A}^2(P)$

$\preceq$ is transitive

$(\forall\beta < \gamma.\forall\alpha < \beta.P^\alpha \preceq P^\beta) \to \forall\beta < \gamma.P^\alpha \preceq \bigcup_{\alpha<\gamma} P^\alpha$

$\forall\alpha < \beta.\mathcal{A}^\alpha \preceq \mathcal{A}^\beta$

Model of extended predicative Mahlo

- Let κ_{M} be a recursively Mahlo ordinal.
- Let κ_{M}^{+} be a recursively inaccessible ordinal above κ_{M}.

$$\Re_{\mathcal{A}^{\kappa_{\mathsf{M}}+1}}(\mathsf{M})$$

$\mathcal{A}^{\kappa_{\mathsf{M}}^{+}}$ is a model of the extended predicative Mahlo universe