Copatterns: Programming Infinite Structures by Observations

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Types 2013 Toulouse, Wed, 24 April 2013

Coalgebras in Functional Programming

- Originally functional programming based on
  - function types,
  - inductive data types.
- In computer science, many computations are interactive.
- Since interactions might go on forever (if not terminated by the user), they correspond to non-wellfounded data types
  - Streams, which are infinite lists,
  - non-wellfounded trees (IO-trees).

Codata Type

- Idea of Codata Types:

$$\begin{array}{l}\text{codata Stream} : \text{Set where} \\ \quad \text{cons} \quad : \quad \mathbb{N} \to \text{Stream} \to \text{Stream}\end{array}$$

- Same definition as inductive data type but we are allowed to have infinite chains of constructors

$$\text{cons}\ n_0\ (\text{cons}\ n_1\ (\text{cons}\ n_2\ \cdots))$$

- **Problem 1:** Non-normalisation.
- **Problem 2:** Equality between streams is equality between all elements, and therefore undecidable.
- **Problem 3:** Underlying assumption is

$$\forall s : \text{Stream}.\exists n, s'.s = \text{cons}\ n\ s'$$

which results in undecidable equality.

Subject Reduction Problem

- In order to repair problem of normalisation restrictions on reductions were introduced.
- Resulted in Coq in a long known problem of **subject reduction**.
- In order to avoid this, in Agda dependent elimination for coalgebras disallowed.
  - Makes it difficult to use.

Problem of Subject reduction:

$$\text{data } \_{==}\_ \ \{A : \text{Set}\}\ (a : A) : A \to \text{Set where}$$
$$\quad \text{refl} : a == a$$

$$\text{codata Stream} : \text{Set where}$$
$$\quad \text{cons} : \mathbb{N} \to \text{Stream} \to \text{Stream}$$

$$\mathsf{zeros} : \text{Stream}$$
$$\mathsf{zeros} = \text{cons } 0 \ \mathsf{zeros}$$

$$\mathsf{force} : \text{Stream} \to \text{Stream}$$
$$\mathsf{force}\ s = \text{case } s \text{ of } (\text{cons } x\ y) \to \text{cons } x\ y$$

$$\mathsf{lem1} : (s : \text{Stream}) \to s == \mathsf{force}(s))$$
$$\mathsf{lem1}\ s = \text{case } s \text{ of } (\text{cons } x\ y) \to \text{refl}$$

$$\mathsf{lem2} : \mathsf{zeros} == \text{cons } 0\ \mathsf{zeros}$$
$$\mathsf{lem2} = \mathsf{lem1}\ \mathsf{zeros}$$
$$\mathsf{lem2} \longrightarrow \text{refl} \ \text{ but } \neg(\text{refl} : \mathsf{zeros} == \text{cons } 0\ \mathsf{zeros})$$

Coalgebraic Formulation of Coalgebras

- Solution is to follow the long established categorical formulation of coalgebras.

Initial F-Algebras

- Inductive data types correspond to initial F-Algebras.
- E.g. the natural numbers can be formulated as

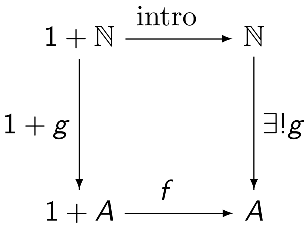
$$
\begin{array}{l}
F(X) = 1 + X \\
\mathrm{intro} : F(\mathbb{N}) \to \mathbb{N} \\
\mathrm{intro}\ (\mathrm{inl}\ *) \quad = \quad 0 \\
\mathrm{intro}\ (\mathrm{inl}\ n) \quad = \quad \mathrm{S}\ n
\end{array}
$$

and we get the diagram

$$
\begin{array}{ccccc}
1 + \mathbb{N} & = & F(\mathbb{N}) & \xrightarrow{\mathrm{intro}} & \mathbb{N} \\
 & 1 + g = F(g) & \downarrow & & \downarrow\ \exists! g \\
1 + A & = & F(A) & \xrightarrow{f} & A
\end{array}
$$

Iteration

Existence of unique g corresponds to unique iteration (example $\mathbb{N}$):



$$
\begin{array}{lllll}
g\ 0 & = & g\ (\mathrm{intro\ inl}) & = & f\ \mathrm{inl} \\
g\ (\mathrm{S}\ n) & = & g\ (\mathrm{intro}\ (\mathrm{inr}\ n)) & = & f\ (\mathrm{inr}\ (g\ n))
\end{array}
$$

By choosing arbitrary f we can define g by pattern matching on its argument n:

$$
\begin{array}{lll}
g\ 0 & = & a_0 \\
g\ (\mathrm{S}\ n) & = & f\ (g\ n) \text{ for some } f : \mathbb{N} \to \mathbb{N}
\end{array}
$$

Recursion and Induction

- From the principle of unique iteration one can derive the principle of recursion:
  Assume

$$\begin{array}{lcl} a_0 & : & A \\ f_0 & : & \mathbb{N} \to A \to A \end{array}$$

  We can then define $g : \mathbb{N} \to A$ s.t.

$$\begin{array}{lcl} g\ 0 & = & a_0 \\ g\ (\mathrm{S}\ n) & = & f_0\ n\ (g\ n) \end{array}$$

- Induction is as recursion but now

$$g : (n : \mathbb{N}) \to A\ n$$

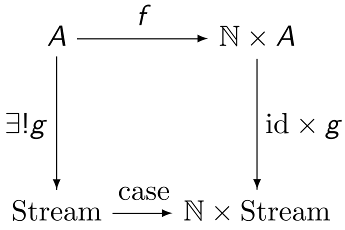
Coalgebras

Final coalgebras F^∞ are obtained by reversing the arrows in the diagram for F-algebras:

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & \mathrm{F}(A) \\ {\scriptstyle \exists! g}\downarrow & & \downarrow{\scriptstyle \mathrm{F}(g)} \\ \mathrm{F}^\infty & \xrightarrow{\;\mathrm{case}\;} & \mathrm{F}(\mathrm{F}^\infty) \end{array}$$

Coalgebras

Consider Streams $= \mathrm{F}^{\infty}$ where $\mathrm{F}(X) = \mathbb{N} \times X$:



Let

$$\text{case } s \quad = \quad \langle \text{head } s, \text{tail } s \rangle$$

and

$$f \ a = \langle f_0 \ a, f_1 \ a \rangle$$

Guarded Recursion

$$
\begin{array}{ccc}
A & \xrightarrow{\langle f_0, f_1 \rangle} & \mathbb{N} \times A \\
\exists! g \downarrow & & \downarrow \mathrm{id} \times g \\
\mathrm{Stream} & \xrightarrow{\langle \mathrm{head}, \mathrm{tail} \rangle} & \mathbb{N} \times \mathrm{Stream}
\end{array}
$$

Resulting equations:

$$
\begin{array}{lcl}
\mathrm{head}\ (g\ a) & = & f_0\ a \\
\mathrm{tail}\ \ (g\ a) & = & g\ (f_1\ a)
\end{array}
$$

Example of Guarded Recursion

$$
\begin{array}{lcl}
\text{head}\ (g\ a) & = & f_0\ a \\
\text{tail}\ \ (g\ a) & = & g\ (f_1\ a)
\end{array}
$$

describes a schema of guarded recursion (or better coiteration)
As an example, with $A = \mathbb{N}$, $f_0\ n = n$, $f_1\ n = n + 1$ we obtain:

$$
\begin{array}{lcl}
\multicolumn{3}{l}{\text{inc} : \mathbb{N} \to \text{Stream}} \\
\text{head}\ (\text{inc}\ n) & = & n \\
\text{tail}\ \ (\text{inc}\ n) & = & \text{inc}\ (n + 1)
\end{array}
$$

Corecursion

In coiteration we need to make in tail always a recursive call:

$$\mathrm{tail}\ (g\ a) = g\ (f_1\ a)$$

Corecursion allows for tail to escape into a previously defined stream. Assume

$$\begin{array}{lll} A & : & \mathrm{Set} \\ f_0 & : & A \to \mathbb{N} \\ f_1 & : & A \to (\mathrm{Stream} + A) \end{array}$$

we get $g : A \to \mathrm{Stream}$ s.t.

$$\begin{array}{llllll} \mathrm{head} & (g\ a) & = & f_0\ a & & \\ \mathrm{tail} & (g\ a) & = & s & \text{if} & f_1\ a = \mathrm{inl}\ s \\ \mathrm{tail} & (g\ a) & = & g\ a' & \text{if} & f_1\ a = \mathrm{inr}\ a' \end{array}$$

Definition of cons by Corecursion

$$\begin{array}{llll} \mathrm{head}\ (g\ a) & = & f_0\ a & \\ \mathrm{tail}\ \ (g\ a) & = & s & \text{if} \quad f_1\ a = \mathrm{inl}\ s \\ \mathrm{tail}\ \ (g\ a) & = & g\ a' & \text{if} \quad f_1\ a = \mathrm{inr}\ a' \end{array}$$

$$\begin{array}{lll} \multicolumn{3}{l}{\mathrm{cons} : \mathbb{N} \to \mathrm{Stream} \to \mathrm{Stream}} \\ \mathrm{head}\ (\mathrm{cons}\ n\ s) & = & n \\ \mathrm{tail}\ \ (\mathrm{cons}\ n\ s) & = & s \end{array}$$

Nested Corecursion

$$
\begin{array}{lcl}
\text{stutter} : \mathbb{N} \to \text{Stream} & & \\
\text{head} \quad\quad (\text{stutter}\ n) & = & n \\
\text{head}\ (\text{tail}\ (\text{stutter}\ n)) & = & n \\
\text{tail} \quad (\text{tail}\ (\text{stutter}\ n)) & = & \text{stutter}\ (n+1)
\end{array}
$$

Even more general schemata can be defined.

Definition of Coalgebras by Observations

- We see now that elements of coalgebras are defined by their observations:
  An element s of Stream is given by defining

$$\begin{array}{lll} \mathrm{head} & s & : & \mathbb{N} \\ \mathrm{tail} & s & : & \mathrm{Stream} \end{array}$$

- This generalises the function type.
  Functions $f : A \to B$ are as well determined by observations, namely by defining

$$f\ a : B$$

  - An $f : A \to B$ is any program which applied to $a : A$ returns some $b : B$.
- **Inductive data types** are defined by **construction** **coalgebraic data types** and **functions** by **observations**.

Relationship to Objects in Object-Oriented Programming

- Objects in Object-Oriented Programming are types which are defined by their observations.
- Therefore objects are coalgebraic types by nature.

Weakly Final Coalgebra

- Equality for final coalgebras is undecidable:
  Two streams

$$\begin{array}{lllllllll} s & = & (a_0 & , & a_1 & , & a_2 & , & \ldots \\ t & = & (b_0 & , & b_1 & , & b_2 & , & \ldots \end{array}$$

  are equal iff $a_i = b_i$ for all i.
- Even the weak assumption

$$\forall s.\exists n, s'.s = \mathrm{cons}\ n\ s'$$

  results in an undecidable equality.
- Weakly final coalgebras obtained by omitting uniqueness of g in diagram for coalgebras.
- However, one can extend schema of coiteration as above, and still preserve decidability of equality.
  - Those schemata are usually not derivable in weakly final coalgebras.

Patterns and Copatterns

- We can define now functions by patterns and copatterns.
- Example define stream:
  $f\ n =$
  $n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f \quad = \quad ? \end{array}$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f \quad = \quad ? \end{array}$$

Pattern match on $f : \mathbb{N} \to \mathrm{Stream}$:

$$\begin{array}{l} f : \mathbb{N} \to \mathrm{Stream} \\ f\ n \quad = \quad ? \end{array}$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{lcl} f & : & \mathbb{N} \to \text{Stream} \\ f\ n & = & ? \end{array}$$

Copattern matching on $f\ n : \text{Stream}$:

$$\begin{array}{lcl} f & : & \mathbb{N} \to \text{Stream} \\ \text{head}\ (f\ n) & = & ? \\ \text{tail}\ \ (f\ n) & = & ? \end{array}$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$\begin{array}{lll} f : \mathbb{N} \to \mathrm{Stream} & & \\ \mathrm{head}\ (f\ n) & = & ? \\ \mathrm{tail}\ \ (f\ n) & = & ? \end{array}$$

Pattern matching on the first $n : \mathbb{N}$:

$$\begin{array}{lll} f : \mathbb{N} \to \mathrm{Stream} & & \\ \mathrm{head}\ (f\ 0) & = & ? \\ \mathrm{head}\ (f\ (\mathrm{S}\ n)) & = & ? \\ \mathrm{tail}\ \ (f\ n) & = & ? \end{array}$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$
\begin{array}{lll}
f : \mathbb{N} \to \mathrm{Stream} & & \\
\mathrm{head}\ (f\ 0) & = & ? \\
\mathrm{head}\ (f\ (\mathrm{S}\ n)) & = & ? \\
\mathrm{tail}\ \ (f\ n) & = & ?
\end{array}
$$

Pattern matching on second $n : \mathbb{N}$:

$$
\begin{array}{lll}
f : \mathbb{N} \to \mathrm{Stream} & & \\
\mathrm{head}\ (f\ 0) & = & ? \\
\mathrm{head}\ (f\ (\mathrm{S}\ n)) & = & ? \\
\mathrm{tail}\ \ (f\ 0) & = & ? \\
\mathrm{tail}\ \ (f\ (\mathrm{S}\ n)) & = & ?
\end{array}
$$

Patterns and Copatterns

$f\ n = n, n, n-1, n-1, \ldots 0, 0, N, N, N-1, N-1, \ldots 0, 0, N, N, N-1, N-1,$

$$
\begin{array}{lcl}
f : \mathbb{N} \to \mathrm{Stream} & & \\
\mathrm{head}\ (f\ 0) & = & ? \\
\mathrm{head}\ (f\ (\mathrm{S}\ n)) & = & ? \\
\mathrm{tail}\ \ (f\ 0) & = & ? \\
\mathrm{tail}\ \ (f\ (\mathrm{S}\ n)) & = & ?
\end{array}
$$

Copattern matching on $\mathrm{tail}\ (f\ 0) : \mathrm{Stream}$

$$
\begin{array}{lcl}
f : \mathbb{N} \to \mathrm{Stream} & & \\
\mathrm{head}\ (f\ 0) & = & ? \\
\mathrm{head}\ (f\ (\mathrm{S}\ n)) & = & ? \\
\mathrm{head}\ (\mathrm{tail}\ (f\ 0)) & = & ? \\
\mathrm{tail}\ (\mathrm{tail}\ (f\ 0)) & = & ? \\
\mathrm{tail}\ (f\ (\mathrm{S}\ n)) & = & ?
\end{array}
$$

Patterns and Copatterns

$$
\begin{array}{l}
f : \mathbb{N} \to \text{Stream} \\
\text{head} \quad (f\ 0\ ) = \ ? \\
\text{head} \quad (f\ (\text{S}\ n)) = \ ? \\
\text{head}\ (\text{tail}\ (f\ 0\ )) = \ ? \\
\text{tail}\ (\text{tail}\ (f\ 0\ )) = \ ? \\
\text{tail} \quad (f\ (\text{S}\ n\ )) = \ ?
\end{array}
$$

Copattern matching on $\text{tail}\ (f\ (\text{S}\ n)) : \text{Stream}$:

$$
\begin{array}{l}
f : \mathbb{N} \to \text{Stream} \\
\text{head} \quad (f\ 0\ ) = \ ? \\
\text{head} \quad (f\ (\text{S}\ n)) = \ ? \\
\text{head}\ (\text{tail}\ (f\ 0\ )) = \ ? \\
\text{tail}\ (\text{tail}\ (f\ 0\ )) = \ ? \\
\text{head}\ (\text{tail}\ (f\ (\text{S}\ n))) = \ ? \\
\text{tail}\ (\text{tail}\ (f\ (\text{S}\ n))) = \ ?
\end{array}
$$

Patterns and Copatterns

We resolve the goals:

$$
\begin{array}{llll}
f : \mathbb{N} \to \mathrm{Stream} \\
\mathrm{head} & (f\ 0) & = & 0 \\
\mathrm{head}\ (\mathrm{tail}\ (f\ 0)) & & = & 0 \\
\mathrm{tail}\ (\mathrm{tail}\ (f\ 0)) & & = & f\ N \\
\mathrm{head} & (f\ (\mathrm{S}\ n)) & = & \mathrm{S}\ n \\
\mathrm{head}\ (\mathrm{tail}\ (f\ (\mathrm{S}\ n))) & & = & \mathrm{S}\ n \\
\mathrm{tail}\ (\mathrm{tail}\ (f\ (\mathrm{S}\ n))) & & = & f\ n
\end{array}
$$

Results of paper in POPL (2013)

- Development of a recursive simply typed calculus (no termination check).
- Allows to derive schemata for pattern/copattern matching.
- Proof that subject reduction holds.

$$t : A, \quad t \longrightarrow t' \text{ implies } t' : A$$

Fibonacci Numbers

Efficient Haskell version adapted to our codata notation:

$$\begin{array}{l} \text{codata Stream} : \text{Set where} \\ \quad \text{cons} : \mathbb{N} \to \text{Stream} \to \text{Stream} \end{array}$$

$$\begin{array}{l} \text{tail} : \text{Stream} \to \text{Stream} \\ \text{tail}\ (\text{cons}\ n\ l) = l \end{array}$$

$$\begin{array}{l} \text{addStream} : \text{Stream} \to \text{Stream} \to \text{Stream} \\ \text{addStream}\ (\text{cons}\ n\ l)\ (\text{cons}\ n'\ l') = \text{cons}\ (n + n')\ (\text{addStream}\ l\ l') \end{array}$$

$$\begin{array}{l} \text{fib} : \text{Stream} \\ \text{fib} = \text{cons}\ 1\ (\text{cons}\ 1\ (\text{addStream fib}\ (\text{tail fib}))) \end{array}$$

Requires lazy evaluation.

Fibonacci Numbers using Coalgebras

$$\begin{array}{l}
\text{coalg Stream} : \text{Set where} \\
\quad \begin{array}{lll} \text{head} & : & \text{Stream} \to \mathbb{N} \\ \text{tail} & : & \text{Stream} \to \text{Stream} \end{array}
\end{array}$$

$$\begin{array}{lll}
\multicolumn{3}{l}{\text{addStream} : \text{Stream} \to \text{Stream} \to \text{Stream}} \\
\text{head (addStream } l\ l') & = & \text{head } l + \text{head } l' \\
\text{tail } \ \text{(addStream } l\ l') & = & \text{addStream (tail } l\text{) (tail } l'\text{)}
\end{array}$$

$$\begin{array}{lll}
\multicolumn{3}{l}{\text{fib} : \text{Stream}} \\
\text{head fib} & = & 1 \\
\text{head (tail fib)} & = & 1 \\
\text{tail } \ \text{(tail fib)} & = & \text{addStream fib (tail fib)}
\end{array}$$

No laziness required. Requires full corecursion (but terminates).

Multiple Constructors in Algebras and Coalgebras

- Having more than one constructor in algebras correspond to disjoint union:

$$
\begin{array}{l}
\mathrm{data}\ \mathbb{N} : \mathrm{Set}\ \ \mathrm{where} \\
\quad \mathsf{0} \quad : \quad \mathbb{N} \\
\quad \mathrm{S} \quad : \quad \mathbb{N} \to \mathbb{N}
\end{array}
$$

  corresponds to

$$
\begin{array}{l}
\mathrm{data}\ \mathbb{N} : \mathrm{Set}\ \ \mathrm{where} \\
\quad \mathrm{intro} \quad : \quad (1 + \mathbb{N}) \to \mathbb{N}
\end{array}
$$

Multiple Constructors in Algebras and Coalgebras

- Dual of disjoint union is products, and therefore multiple destructors correspond to product:

$$\begin{array}{l} \text{coalg Stream} : \text{Set where} \\ \quad \begin{array}{lll} \text{head} & : & \text{Stream} \to \mathbb{N} \\ \text{tail} & : & \text{Stream} \to \text{Stream} \end{array} \end{array}$$

corresponds to

$$\begin{array}{l} \text{coalg Stream} : \text{Set where} \\ \quad \begin{array}{lll} \text{case} & : & \text{Stream} \to (\mathbb{N} \times \text{Stream}) \end{array} \end{array}$$

Codata Types Correspond to Disjoint Union

- Consider

$$\begin{array}{l} \text{codata coList} : \text{Set where} \\ \quad \begin{array}{lcl} \text{nil} & : & \text{coList} \\ \text{cons} & : & \mathbb{N} \to \text{coList} \to \text{coList} \end{array} \end{array}$$

- Cannot be simulated by a coalgebra with several destructors.

Simulating Codata Types by Simultaneous Algebras/Coalgebras

- Represent Codata as follows

$$
\begin{array}{l}
\textsf{mutual} \\
\quad \text{coalg coList} : \text{Set where} \\
\quad\quad \text{unfold} : \text{coList} \to \text{coListShape} \\
\\
\quad \text{data coListShape} : \text{Set where} \\
\quad\quad \text{nil} \quad : \quad \text{coListShape} \\
\quad\quad \text{cons} \quad : \quad \mathbb{N} \to \text{coList} \to \text{coListShape}
\end{array}
$$

Definition of Append

$$\text{append} : \text{coList} \to \text{coList} \to \text{coList}$$
$$\text{append}\ l\ l' = ?$$

Definition of Append

$$\begin{array}{l} \text{append} : \text{coList} \to \text{coList} \to \text{coList} \\ \text{append}\ l\ l' = ? \end{array}$$

We copattern match on $\text{append}\ l\ l' : \text{coList}$:

$$\begin{array}{l} \text{append} : \text{coList} \to \text{coList} \to \text{coList} \\ \text{unfold}\ (\text{append}\ l\ l') = ? \end{array}$$

Definition of Append

$$\text{append} : \text{coList} \to \text{coList} \to \text{coList}$$
$$\text{unfold}\ (\text{append}\ l\ l') = ?$$

We cannot pattern match on l.
But we can do so on $(\text{unfold}\ l)$:

$$\begin{array}{l}
\text{append} : \text{coList} \to \text{coList} \to \text{coList} \\
\text{unfold}\ (\text{append}\ l\ l') = \\
\quad \text{case}\ (\text{unfold}\ l)\ \text{of} \\
\qquad \begin{array}{lcl} \text{nil} & \to & ? \\ (\text{cons}\ n\ l) & \to & ? \end{array}
\end{array}$$

Definition of Append

$$
\begin{array}{l}
\text{append} : \text{coList} \to \text{coList} \to \text{coList} \\
\text{unfold}\ (\text{append}\ l\ l') = \\
\quad \text{case}\ (\text{unfold}\ l)\ \text{of} \\
\qquad \begin{array}{lcl}
\text{nil} & \to & ? \\
(\text{cons}\ n\ l) & \to & ?
\end{array}
\end{array}
$$

We resolve the goals:

$$
\begin{array}{l}
\text{append} : \text{coList} \to \text{coList} \to \text{coList} \\
\text{unfold}\ (\text{append}\ l\ l') = \\
\quad \text{case}\ (\text{unfold}\ l)\ \text{of} \\
\qquad \begin{array}{lcl}
\text{nil} & \to & \text{unfold}\ l' \\
(\text{cons}\ n\ l) & \to & \text{cons}\ n\ (\text{append}\ l\ l')
\end{array}
\end{array}
$$

Conclusion

- Symmetry between
  - algebras and coalgebras,
  - iteration and coiteration,
  - recursion and corecursion,
  - patterns and copatterns.
- Final algebras are defined by constuction,
  coalegbras and function types by observation.
- Codata construct assumes every element is introduced by a constructor, which results in
  - either undecidable equality
  - or requires sophisticated restrictions on reduction rule which are difficult to get right.
    - Problem of subreduction in Coq.
    - Too restrictive elimination principle in Agda.
- Weakly final coalgebras solve this problem, by having reduction rules which can always be applied independent of context.

More Details

- More details can be found in my proper talk tomorrow.
  - Assumption $\forall s : \mathrm{Stream}.\exists n, s'.s = \mathrm{cons}\ n\ s'$ results in undecidable equality.
  - How to replace copattern matching by combinators.