Inductive-inductive definitions

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Induction is a powerful and important principle of definition, perhaps especially so in dependent type theory and constructive mathematics. However, many rather natural constructions cannot be introduced by ordinary inductive definitions alone, such as lists of a certain length or universes in type theory. This leads us to consider variants such as indexed inductive definitions, where a family of sets is simultaneously inductively defined, or inductive-recursive definitions, where a set U is inductively defined together with a recursive function $T : U \to D$ for some fixed (large) type D.

In this talk, we will introduce yet another variant, called *induction-induction*, which generalises indexed inductive definitions. Induction-induction gives the possibility to simultaneously introduce a set A together with an A-indexed set B, i.e. for every $a : A$, $B(a)$ is a set. Both A and $B(a)$ are inductively defined, and the constructors for A can also refer to B and vice versa.

Instances of induction-induction have implicitly been used by several researchers (Dybjer, Danielsson, Chapman) to model type theory inside type theory, and inductive-inductive definitions are available in the proof assistant/ programming language Agda, but without justification of the principle.