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Domain representations of topological spaces (Extended abstract)

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Abstract

A *domain representation* of a topological space X is a function, usually a quotient map, from a subset of a domain onto X.

Several different classes of domain representations are introduced and studied. It is investigated when it is possible to build domain representations from existing ones. It is, for example, discussed whether there exists a natural way to build a domain representation of a product of topological spaces from given domain representations of the factors. It is shown that any topological space has a domain representation. These domain representations are very large. However, smaller domain representations are also constructed for large classes of spaces. For example, each second countable regular Hausdorff space has a domain representation with a countable base. Domain representations of functions and function spaces are also studied.

Key words: Domain theory; Topological spaces; Representations; Computability.

1 Introduction

In this paper we study domain representations of topological spaces and properties of such representations. The main reason for studying such representations is that they provide a uniform method to introduce computability on abstract spaces such as $\mathbb{R}$. Scott–Ershov domains carry a natural computability theory and the representing map from the domain onto the topological space imports the computability theory onto the topological space. We will in this paper not directly concern ourselves with computability but will instead study the notion of domain representability abstractly. This paper is an extended abstract of the forthcoming paper [7]. Most results herein also appear in [6, Chapter 4].

The notion of domain representations is introduced in Section 3. The domain representations are classified depending on the properties of the representation. They are primarily classified by the topological properties of the representing function. Several other useful properties that a representation may or may not have are also identified.

Neighbourhood systems are introduced in Section 4. They are used to construct domain representations with many good properties. In particular, it is shown that any regular Hausdorff space has an upwards-closed retract representation. The neighbourhood system chosen in this case consists of all the non-empty closed sets of the space. If the space is second countable, then a modification of the neighbourhood system gives a domain representation with a countable base. Furthermore, it is shown that spaces with upwards-closed retract representations are regular Hausdorff spaces. Hence we have a complete characterisation of the spaces that have an upwards-closed retract representation.

Domain representations where the representing elements are a subset of the maximal elements of the domain are constructed in Section 5. It is possible to make such constructions for arbitrary spaces. However, these representations are very large and lack some of the properties that the representations constructed in Section 4 possess.

In Section 6 we study when it is possible to uniformly build domain representations for spaces obtained by a topological construction from domain representations of the old spaces. We have, for example, that retract representations are uniformly closed under retracts, subspaces, disjoint unions, and products.

Domain representations of functions are studied in Section 7.1. A function has a domain representation if there exists a domain function inducing it. Domain functions satisfying a natural condition always induce a continuous function on the represented spaces. Theorem 7.3 gives sufficient conditions so that representations of functions always exist.

Section 7.2 studies when the function space construction on domains can be used to give domain representations of function spaces. A representation of a function space induces a topology on the function space. Under some conditions this topology is proven (Theorem 7.6) to be exactly the compact-open topology.

2 Domain-theoretic background

In this subsection we will briefly review domain theory. We concentrate on giving the notions and hint at some results. The proofs are generally omitted and can be found in either [21] or [1].

Let $D = (D, \sqsubseteq)$ be a partially ordered set. A subset $A \subseteq D$ is an *upper set* if $x \in A$ and $x \sqsubseteq y$ implies $y \in A$. Let $\uparrow A = \{y \in D : \exists x \in A(x \sqsubseteq y)\}$. We will abbreviate $\uparrow\{x\}$ by $\uparrow x$. The dual notions are *lower set* and $\downarrow A$. A subset $A \subseteq D$ is *directed* if $A \neq \emptyset$ and whenever $x, y \in A$ then there is $z \in A$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$. The supremum, or least upper bound, of A (if it exists) is denoted by $\bigsqcup A$.

A *complete partial order*, abbreviated *cpo*, is a partial order, $D = (D; \sqsubseteq, \bot)$, such that $\bot$ is the least element in D and any directed set $A \subseteq D$ has a supremum, $\bigsqcup A$. This is also known as a pointed *dcpo* in the literature.

Note that our definition of cpo includes a bottom element. The existence of bottom elements is useful in, e.g., function space constructions. The more general form, without bottom, is not needed in our work. In addition it is intuitively pleasing to have a bottom element since this will correspond to the trivial approximation of a point in a topological space. That is, the bottom element approximates the whole space.

Let D be a cpo. Then an element $a \in D$ is *compact* if whenever $A \subseteq D$ is a directed set and $a \sqsubseteq \bigsqcup A$, then $a \in \downarrow A$. The set of compact elements of D is denoted by D_c.

A cpo D is *algebraic* if for each $x \in D$, the set $\text{approx}(x) = \downarrow x \cap D_c$ is directed and $x = \bigsqcup \text{approx}(x)$. A cpo D is *consistently complete* if $\bigsqcup A$ exists in D whenever $A \subseteq D$ is a consistent set, i.e., has an upper bound.

Definition 2.1 A *Scott–Ershov domain*, or simply *domain*, is a consistently complete algebraic cpo.

The topology normally used on domains is called the Scott topology. Let D be an algebraic cpo. A subset U of D is open if

(i) U is an upper set, and

(ii) $x \in U$ implies that there exists $a \in \text{approx}(x)$ such that $a \in U$.

An easy observation is that the Scott topology on a domain is T_0. However the Scott topology fails to be T_1 on all domains except the trivial domain consisting of a single element.

The sets $\uparrow a$, for $a \in D_c$, constitute a base for the Scott topology on a domain D.

Let D and E be domains. A function $f: D \to E$ is Scott continuous if f is monotone and

$$f(\bigsqcup A) = \bigsqcup f[A],$$

for any directed $A \subseteq D$. The notion of Scott continuity coincides with the notion of continuity induced from the Scott topology on the domains.

Any continuous function between domains is determined by its values on the compact elements.

Let D and E be domains and let $f: D_c \to E$ be a monotone function. Then there exists a unique extension of $g: D \to E$ of f such that $f = g|_{D_c}$.

The function space $[D \to E]$ consists of all continuous functions from the domain D into the domain E. For $a \in D_c$ and $b \in E_c$ the step function $\langle a; b\rangle$

defined by

$$\langle a; b\rangle(x) = \begin{cases} b, & \text{if } a \sqsubseteq x; \\ \bot, & \text{otherwise.} \end{cases}$$

is a continuous function. The compact elements of the function space are finite suprema of consistent sets of such step functions.

Domains are often constructed as the completion of some underlying structure. We will study the type of structure from which we can construct Scott–Ershov domains.

The compact elements D_c of a Scott–Ershov domain D form a conditional upper semilattice with least element, abbreviated *cusl*. That is, a cusl is a partially ordered set where a least upper bound exists for every pair of elements that have an upper bound.

An *ideal* is a directed lower set. The ideal completion over a cusl P is the set of all ideals over P, denoted $\mathrm{Idl}(P)$. When ordered by set inclusion the ideal completion of a cusl forms a Scott–Ershov domain. For a in a cusl P, $\downarrow a$ is an ideal, the *principal ideal* generated by a. The compact elements of $\mathrm{Idl}(P)$ are the principal ideals $\downarrow a$, for $a \in P$.

The representation theorem for Scott–Ershov domains tells us that any Scott–Ershov domain is the ideal completion of a cusl.

Theorem 2.2 *Let D be a Scott–Ershov domain. Then* $\mathrm{Idl}(D_c) \cong D$.

We clearly have the following equivalence, for $I \in \mathrm{Idl}(P)$

$$\downarrow a \subseteq I \iff a \in I.$$

Thus the sets $B_a = \{I \in \mathrm{Idl}(P) : a \in I\}$ for $a \in P$ form a base for the Scott topology on $\mathrm{Idl}(P)$.

3 Domain representations

Representations of topological spaces by domains or embeddings of topological spaces into domains have been studied by several people. Weihrauch and Schreiber [27] considered embeddings of metric spaces into cpos with weight and distance. Stoltenberg-Hansen and Tucker [22,24] introduced the notion of domain representability. Edalat [8–10] has used embeddings into continuous dcpos to study integration, measures and fractals. Edalat and Heckmann [11] and di Gianantonio [15] among others have also studied similar notions. Ershov's [12] representation of the Kleene–Kreisel continuous functionals is an early example of a domain representation.

In a domain representation D of a space we isolate the set of representing elements D^{R} as those that contain *total* or *complete* information. This has led to the abstract study of domains with totality, i.e., domains with a distinguished subset of total elements. This sort of study has been pursued in connection with certain type structures by Berger [3,4], Kristiansen and Normann [17], Normann [18,20] and Waagbø [26].

The kind of representations or embeddings that are possible for a certain topological space are affected by the choice of domains. For example, any metric space can be embedded into the maximal elements of a continuous dcpo. For Scott–Ershov domains we know that the set of maximal elements is Hausdorff and has a clopen base and hence that any space embedded into the maximal elements of a Scott–Ershov domain must be totally disconnected.

We only consider domain representations by Scott–Ershov domains here. This is due to Scott–Ershov domains having a simpler computability theory (not exploited here, however) and, in our experience, sufficiency in terms of representability. Sufficiency can to some extent be motivated by [7], where it is shown that any consistently complete continuous cpo has a dense retract representation.

3.1 Classes of quotient maps

The primary classification of our domain representations will be the topological properties of the representing function. We introduce here the different classes of quotient maps that we will consider.

Definition 3.1 Let $f\colon X \to Y$ be a continuous function between the topological spaces X and Y. Then

(i) f is a *quotient map* if $V \subseteq Y$ is open if, and only if, $f^{-1}[V]$ is open,

(ii) f is *pseudo-open* if for any $y \in Y$ and any open set $U \subseteq X$ containing $f^{-1}[y]$, y is in the interior of $f[U]$,

(iii) f is *open* if $f[U]$ is open for any open subset U, and

(iv) f is a *retract* if there exists $e\colon Y \to X$ such that $f \circ e = \mathrm{id}_Y$.

The notion of pseudo-open is due to Arhangelskij [2].

Remember that if $f\colon X \to Y$ is a quotient then $X/{\sim}$ and the image $f[X]$ are homeomorphic, where $\sim$ is the equivalence relation induced by f.

We observe some relationships between the introduced classes. Onto open maps are pseudo-open and onto pseudo-open maps are quotients. Retracts are pseudo-open. The notions of open and retract are independent from each other. Moreover, the classes of quotient maps introduced above are all closed under composition.

3.2 Classes of domain representations

We now give the fundamental definition of domain representability. The notion is a stronger version of the one that appears in [22,24].

Definition 3.2 Let D be a domain, $D^{\mathrm{R}} \subseteq D$, and let X be a topological space. Suppose that φ is a continuous mapping from D^{R} onto X. We call the triple $(D, D^{\mathrm{R}}, \varphi)$

(i) a *weak domain representation* of X;

(ii) a *domain representation* of X if φ is a quotient map;

(iii) a *pseudo-open domain representation* of X if φ is pseudo-open;

(iv) an *open domain representation* of X if φ is open;

(v) a *retract domain representation* of X if φ is a retract;

(vi) a *homeomorphic domain representation* of X if φ is a homeomorphism.

We will often drop the word domain from the notions above since we only consider domain representations.

The set D^{R} above will be called the set of representing elements. The representing domain D contains both proper approximations and total or complete representations of elements of X, the latter constituting the set D^{R}. Intuitively, D^{R} consists of those domain elements which contain sufficient information to completely determine an element in X via φ.

Each of the introduced classes of domain representations implies the earlier ones, with the exception that retract representations does not imply open representation. However, every retract representation $(D, D^{\mathrm{R}}, \varphi)$ of X with embedding η induces a homeomorphic (and hence open) representation $(D, \eta^{-1}[X], \varphi)$ of X.

If $(D, D^{\mathrm{R}}, \varphi)$ is a homeomorphic representation of X, then φ^{-1} is an embedding of X into D.

There are other criteria for suitability of a representation, besides the topological properties of the representing function φ, namely the kind of the domain D and the properties of the set D^{R}.

Definition 3.3 Let $(D, D^{\mathrm{R}}, \varphi)$ be a domain representation. The representation is *upwards-closed* if $d \in D^{\mathrm{R}}$ and $d \sqsubseteq e$ implies $e \in D^{\mathrm{R}}$ and $\varphi(e) = \varphi(d)$.

If the represented space X in the definition above is T_1, then it is redundant to require $\varphi(e) = \varphi(d)$.

In a natural representation we would like to consider all elements below a domain element as approximations to the point that that element represents. In this setting, the representing elements will be *total* or *complete* in the sense that they contain total information about the point they represent. Since any element above a representing element contains more information we clearly see that any natural representation should be upwards-closed.

Let $(D, D^{\mathrm{R}}, \varphi)$ be a domain representation. If every element of D^{R} is a maximal element, then we say that it is a *representation by maximal elements* and note that it is upwards-closed. However, we have noted that only totally disconnected spaces can be given an upwards-closed homeomorphic domain representation by maximal elements. We can construct upwards-closed domain representations of a large class of spaces if we content ourselves with representing elements that are sufficiently high up in the domain so that no contradictory information can appear above them. More formally, we require only that each representing element is *total* in the sense that $\uparrow x$ is directed. We say that we have a *representation by total elements*.

A *dense representation* is a representation $(D, D^{\mathrm{R}}, \varphi)$ where D^{R} is dense in D. In Section 7.1 denseness is used to show that functions can be represented by (or lifted to) domain functions.

Many of our representations satisfies that for every $x \in X$ there exists a least representative d_x of x, or equivalently, $\varphi^{-1}[x]$ has a least element. Clearly, any homeomorphic representation has this property. For a representations satisfying this property we say that it is a *representation with least representatives*.

A domain representation is said to have the *closed image property* if $\varphi[\uparrow a \cap D^{\mathrm{R}}]$ is closed for all $a \in D_c$.

4 Standard representations by domains of filter bases

4.1 Neighbourhood systems

This subsection introduces the notion of neighbourhood systems. These structures will be used in the subsequent subsections to construct domain representations.

The interior and closure of a subset $A \subseteq X$ are denoted by A° and $\overline{A}$ respectively.

Definition 4.1 Let X be a topological space and let P be a family of non-empty subsets of X such that $X \in P$. Then $P = (P; \supseteq, X)$ is a *neighbourhood system* if the following are satisfied:

(i) if $A, A' \in P$ and $A \cap A' \neq \emptyset$ then $A \cap A' \in P$, and

(ii) if $x \in U$, where U is open, then $(\exists A \in P)(x \in A^\circ \subseteq A \subseteq U)$.

Examples of neighbourhood systems are: the non-empty closed sets of a regular space; the non-empty compact sets of a locally compact space; and all non-empty sets of a base for the topology together with the set X. The former two may be called *closed* neighbourhood systems and the latter an *open* neighbourhood system.

Condition (i) makes P ordered with reverse inclusion into a cusl. Hence the ideal completion $D = \mathrm{Idl}(P)$ is a domain. The elements of D are ideals in $(P, \supseteq)$, i.e., they are *filter bases* in the topological sense. The Scott topology on D is generated by the basic open sets $\uparrow\downarrow A = B_A = \{I \in D : A \in I\}$ for $A \in P$.

The elements of P may be seen as approximations of elements of X. These approximations are consistent if they have a non-empty intersection. P is an approximation for X in the sense of [24].

For each element of the space X we define two ideals of special interest.

Definition 4.2 Let P be a neighbourhood system for X and let $x \in X$.

(i) $I_x = \{A \in P : x \in A^\circ\}$.

(ii) $J_x = \{A \in P : x \in A\}$.

Clearly, $I_x \subseteq J_x$, and if P is an open neighbourhood system then $I_x = J_x$. For any $A \in P$ there exists $x \in A$. Clearly, $J_x \in B_A$. Thus, the set $\{J_x : x \in X\}$ is dense in D.

Define $\eta\colon X \to D$ by $\eta(x) = I_x$.

Lemma 4.3 (i) *The function η is continuous.*

(ii) *If η is injective, then η is an embedding of X into D.*

Proof. (i) $\eta(x) \in B_A \iff A \in I_x \iff x \in A^\circ$.

(ii) It is sufficient to show that $\eta\colon X \to \eta[X]$ is open. We leave to the reader to verify that if U is an open set then

$$\eta[U] = \eta[X] \cap \bigcup_{A \subseteq U} B_A.$$

□

An ideal I *converges* to a point $x \in X$, denoted $I \to x$, if for every open set U containing x, there is an $A \in I$ such that $x \in A \subseteq U$, or equivalently, if the filter base corresponding to I converges to x. We note that $I \to x$ if, and only if, $I_x \subseteq I$.

4.2 Homeomorphic representations for T_0-spaces

Theorem 4.4 *Any T_0-space X has a dense homeomorphic representation.*

Proof. Let P be a neighbourhood system consisting of all the non-empty sets of a base for the topology together with the set X.

Due to the T_0-property of X, the function $\eta\colon X \to D$ defined as above is injective. Hence, by Lemma 4.3, η is an embedding of X into D. Thus, $(D, \eta[X], \eta^{-1})$ is a homeomorphic representation of X.

Since P is an open neighbourhood system we have $I_x = J_x$, and hence $D^{\mathrm{R}} = \eta[X]$ is dense in D. □

In general the representation above is not upwards-closed and not by maximal or total elements as the following example shows.

Example 4.5 The ideals I_x need not be maximal, in fact they need not even be total. Suppose $X = \mathbb{R}$ and let P consist of all non-empty open intervals. Then I_x consists of all open intervals containing x. Let I_0^+ and I_0^- be the ideals generated by $I_0 \cup \{(0,1)\}$ and $I_0 \cup \{(-1,0)\}$ respectively. The ideals I_0^+ and I_0^- are not consistent, i.e., I_0 is not total.

Theorem 4.6 *A space X with a retract representation $(D, D^{\mathrm{R}}, \varphi, \eta)$ is a T_0-space.*

Proof. If x and x' are inseparable by open sets, then the same holds true for $\eta(x)$ and $\eta(x')$ since η is an embedding. Hence, $\eta(x) = \eta(x')$ since domains are T_0. Applying φ we have $\varphi(\eta(x)) = \varphi(\eta(x'))$, i.e., $x = x'$. □

4.3 Upwards-closed retract representations for regular Hausdorff spaces

Let P be a neighbourhood system for a Hausdorff space X and let $D = \mathrm{Idl}(P)$. Let D^{R} be the set of converging ideals. The Hausdorff property implies that every converging ideal has a unique limit point. Define $\varphi: D^{\mathrm{R}} \to X$ by mapping a converging ideal to its limit point.

Let $x \in X$ and $A \in P$. By the properties of a neighbourhood system, $x \in \overline{A}$ if, and only if, there exists an ideal I containing A and converging to x. Thus, φ will have the closed image property since $\varphi[B_A \cap D^{\mathrm{R}}] = \overline{A}$.

It is clear that φ is onto and that the representation will be upwards-closed. However, in order to show continuity of φ, we need to strengthen (ii) in Definition 4.1 to:

(ii)$'$ if $x \in U$, where U is open, then $(\exists A \in P)(x \in A^{\circ} \subseteq \overline{A} \subseteq U)$.

If the neighbourhood system P satisfies (ii)$'$, then φ is continuous.

Summarising we have the following theorem.

Theorem 4.7 *Any regular Hausdorff space X has a dense upwards-closed retract representation by total elements with least representatives and closed image property.*

Proof. Choose for example P to consist of all non-empty closed sets of the space. That a converging ideal is total is due to J_x being the maximal ideal converging to x. Clearly, I_x is the least ideal representing x. □

By restricting D^{R} to $\eta[X]$ we get a homeomorphic representation by total elements of X. However, this representation is neither dense nor upwards-closed in general, and it may also fail the closed image property.

Let X be a second countable regular Hausdorff space and let B be a countable base for the topology on X. Let P consist of all finite non-empty intersections of sets in $\{\overline{U} : U \in B\}$. Then P satisfies the stronger version of neighbourhood systems. Note that P is countable, i.e., the constructed domain D has a countable base.

Clearly, Theorem 4.7 does not yield an effective domain in general. However, for the reals $\mathbb{R}$ it is possible to construct an effective domain by choosing the neighbourhood system to consist of the rational intervals. In [5] a general construction of effective domain representations of metric spaces is given.

Theorem 4.8 *Let $(D, D^{\mathrm{R}}, \varphi, \eta)$ be an upwards-closed retract representation of X. Then X is a regular Hausdorff space.*

Proof. Let x_1 and x_2 be distinct points in X. Then $\eta(x_1)$ and $\eta(x_2)$ are inconsistent by upwards-closed. Hence, there exist disjoint open sets U_i such

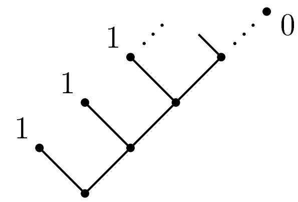

Fig. 1. Domain D representing the Sierpinski space.

that $\eta(x_i) \in U_i$. Thus, $\eta^{-1}[U_i]$ are disjoint open sets containing x_i.

Let x be in the open set U. Then $\eta(x) \in \varphi^{-1}[U]$. There exists $a \in D_c$ such that $\eta(x) \in \uparrow a \subseteq \varphi^{-1}[U]$.

$$\begin{aligned}
x &\in \eta^{-1}[\uparrow a \cap D^{\mathrm{R}}] \quad \text{(open)} \\
&\subseteq \eta^{-1}[\downarrow\uparrow a \cap D^{\mathrm{R}}] \quad \text{(closed)} \\
&\subseteq \varphi[\downarrow\uparrow a \cap D^{\mathrm{R}}] \\
&= \varphi[\uparrow a \cap D^{\mathrm{R}}] \quad \text{(by upwards-closed)} \\
&\subseteq \varphi[\varphi^{-1}[U]] \\
&= U.
\end{aligned}$$

□

Corollary 4.9 *The spaces with upwards-closed retract representations are exactly the regular Hausdorff spaces.*

It is not possible to drop the requirement of retract in the theorem above as the following example shows.

Example 4.10 Let X be the Sierpinski space, i.e., $X = \{0, 1\}$ and the topology on X is $\{\emptyset, \{1\}, X\}$. The Sierpinski space is T_1, but not Hausdorff. However, we can give an upwards-closed open representation of the Sierpinski space.

Build a domain D as in Figure 1. Let D^{R} be the set of maximal elements of D. Define φ as indicated in the figure, i.e., the only non-compact element is mapped to 0, the rest of the maximal elements are mapped to 1.

Clearly, $(D, D^{\mathrm{R}}, \varphi)$ represents X. It is upwards-closed since D^{R} consists of maximal elements. Any basic open set in D contains a maximal element that represents 1. Hence, the forward image of any basic open set is open. Thus, the representation is open.

5 Representations by maximal elements

Any space can be represented by maximal elements. However, we will give the construction only for a simpler case.

A topological space X is *sequential* if a set $A \subseteq X$ is open if and only if every sequence $(x_n)_n$ converging to a point in A is eventually in A.

We give a direct construction of a domain representation of a sequential space. In the proof below let s denote a sequence whose elements are s_n for

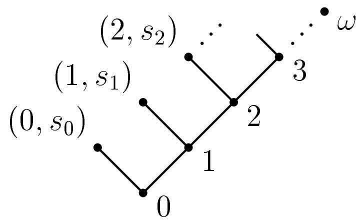

Fig. 2. The domain $D_{s,x}$.

$n \in \mathbb{N}$.

Theorem 5.1 *Any sequential space X has a representation by maximal elements.*

Proof. Let $\mathcal{S} = \{(s, x) : s \in X^{\mathbb{N}}, x \in X,$ and s converges to $x\}$. For each $(s, x) \in \mathcal{S}$ construct a domain $D_{s,x}$, see Figure 2, whose compact elements are

$$\{(n, s_n) : n \in \mathbb{N}\} \cup \mathbb{N}.$$

The ordering on $D_{s,x}$ is the one suggested by the figure.

Let D be the separated sum of $\{D_{s,x} : (s, x) \in \mathcal{S}\}$. The maximal elements of D is the set $D_m = \{(n, s_n)_{D_{s,x}} : n \in \mathbb{N},\ (s, x) \in \mathcal{S}\} \cup \{\omega_{D_{s,x}} : (s, x) \in \mathcal{S}\}$. Define a function $\varphi \colon D_m \to X$ by

$$\varphi(y) = \begin{cases} s_n, \text{ if } y = (n, s_n)_{D_{s,x}}; \\ x, \ \text{ if } y = \omega_{D_{s,x}}. \end{cases}$$

To show that φ is continuous it is sufficient to show that φ is continuous on each $D_{s,x} \cap D_m$. Let $U \subseteq X$ be open and let $V = \varphi^{-1}[U] \cap D_{s,x}$. Clearly any point of the form $(n, s_n) \in V$ is interior since (n, s_n) is compact in $D_{s,x}$. If $\omega \in V$ then there must exist an n such that $s_m \in U$ for all $m \geq n$ since s converges to x in the open set U. Hence $\{y \in D_{s,x} : n \sqsubseteq y\} \cap D_m$ is an open set of D_m included in V, i.e., x is an interior point of V. Thus φ is continuous on every $D_{s,x} \cap D_m$ and hence on D_m.

Suppose $A \subseteq X$ and $\varphi^{-1}[A]$ is open in D_m. If $s \to x \in A$ then $\omega_{D_{s,x}} \in \varphi^{-1}[A]$ which is open in D_m, hence there exists an open set $\{y : n_{D_{s,x}} \sqsubseteq y\} \cap D_m \subseteq \varphi^{-1}[A]$ such that $\omega_{D_{s,x}} \in \{y : n_{D_{s,x}} \sqsubseteq y\} \cap D_m$. Thus the sequence s is eventually in A. Since any sequence converging to a point of A eventually is in A and X is a sequential space we have that A is open. We have shown that φ is a quotient map. Hence (D, D_m, φ) is a representation of X. □

The domain D constructed in the proof above has an uncountable base in general.

We can generalise the above construction to show that any topological space has a representation by maximal elements. The generalisation uses nets instead of sequences to build the domain.

Theorem 5.2 *Any topological space X has a representation by maximal elements.*

Moreover, it is easy to lift a continuous function to representations of this kind even though it is not covered by Theorem 7.3.

6 Uniform closure properties

Now we study uniform closure properties of representations under topological constructions of spaces. Thus we are interested in when we can construct a new representation of the newly created space in a canonical way from the representation(s) of the old space(s). All constructions made in this section preserves effectivity.

6.1 Open and pseudo-open images and quotients

Given a representation of a space we can represent certain images of that space with the same domain and with the same set of representing elements by composition of the representing function and the image map.

Proposition 6.1 *Representability, pseudo-open representability, retract representability and open representability is uniformly closed under quotients, pseudo-open images, retracts and open images, respectively.*

Proof. The classes of quotient maps introduced in Section 3 are all closed under composition. □

6.2 Subspaces

Given a representation $(D, D^{\mathrm{R}}, \varphi)$ of a space X, we are interested in when D with φ restricted to the inverse image of a subset $Y \subseteq X$ is a representation of Y. The following result is easily obtained.

Proposition 6.2 *Let $(D, D^{\mathrm{R}}_X, \varphi)$ be a weak representation of X and let Y be a subset of X, and let $D^{\mathrm{R}}_Y = \varphi^{-1}[Y]$.*

(i) *If $(D, D^{\mathrm{R}}_X, \varphi)$ is a weak, pseudo-open, open, retract or homeomorphic representation, then $(D, D^{\mathrm{R}}_Y, \varphi|_{D^{\mathrm{R}}_Y})$ is a representation of the same kind of Y.*

(ii) *If $(D, D^{\mathrm{R}}_X, \varphi)$ is a representation and Y is either an open or a closed subset of X, then $(D, D^{\mathrm{R}}_Y, \varphi|_{D^{\mathrm{R}}_Y})$ is a representation of Y.*

The following example shows that (ii) in the proposition above cannot be strengthened to an arbitrary subset Y of X, in fact, it does not hold for G_δ subsets.

Example 6.3 Let Z be the following subset of the euclidean plane; $Z = \{(x, 0) : 0 \neq x \in \mathbb{R}\} \cup \{(\frac{1}{n}, 1) : 0 < n \in \mathbb{N}\} \cup \{(0, 1)\}$. Let $(D, D^{\mathrm{R}}, \varphi)$ be a retract representation of $\mathbb{R}^2$ constructed as in Theorem 4.7. By Proposition 6.2 (i) $(D, D^{\mathrm{R}}_Z, \varphi)$, where $D^{\mathrm{R}}_Z = \varphi^{-1}[Z]$ is a retract representation of Z. Let X be the projection of Z onto its first coordinate. If p is the projection then

$(D, D^{\mathrm{R}}_Z, p \circ \varphi)$ is a representation of X since p is a quotient. Let $Y = X \setminus \{\frac{1}{n} : 0 < n \in \mathbb{N}\}$ and let $D^{\mathrm{R}}_Y = D^{\mathrm{R}}_Z \cap \varphi^{-1}[p^{-1}[Y]]$. Then $(D, D^{\mathrm{R}}_Y, p \circ \varphi|_{D^{\mathrm{R}}_Y})$ is a weak domain representation of Y.

Let U be the open ball centred in $(0,1)$ with radius $\frac{1}{2}$. By the construction of D there is $\downarrow A \in D_c$ such that $(0,1) \in A^\circ \subseteq A \subseteq U$. Clearly, if $I \in B_A \cap D^{\mathrm{R}}_Y$ then $\varphi(I) = (0,1)$, i.e., $p(\varphi(I)) = 0$. Thus $B_A \cap D^{\mathrm{R}}_Y \subseteq \varphi^{-1}[p^{-1}[0]]$. On the other hand, if $I \in \varphi^{-1}[p^{-1}[0]]$ then $\varphi(I) = (0,1)$, and hence, $I \in B_A$. We have shown $B_A \cap D^{\mathrm{R}}_Y = \varphi^{-1}[p^{-1}[0]]$. It follows that $p \circ \varphi|_{D^{\mathrm{R}}_Y}$ is not a quotient since $\{0\}$ is not an open set in Y.

6.3 Disjoint sums and direct limits

We now briefly consider disjoint sums and direct limits of topological spaces.

Proposition 6.4 *Weak representation, representability, pseudo-open representability, retract representability and open representability are uniformly closed under disjoint topological sum.*

Proof. Let X be the disjoint topological sum of $\{X_i : i \in I\}$ and suppose $(D_i, D^{\mathrm{R}}_i, \varphi_i)$ are weak representations of X_i. Let D be the separated sum of $\{D_i : i \in I\}$. Clearly $(D, \biguplus_i D^{\mathrm{R}}_i, \biguplus_i \varphi_i)$ is a weak representation of X.

Observe that $\biguplus_i \varphi_i$ has the required property if each φ_i has that property. □

Proposition 6.5 *Weak representability and representability are uniformly closed under direct limits.*

Proof. By Proposition 6.1 and Proposition 6.4 using the standard construction of a direct limit as a quotient of a disjoint sum. □

6.4 Products

In this subsection we consider uniform representations of cartesian products. The situation here is, perhaps surprisingly, somewhat problematic. For weak representations, however, it is straightforward.

Proposition 6.6 *For every $i \in I$ let $(D_i, D^{\mathrm{R}}_i, \varphi_i)$ be a weak representation of X_i. Then $(\prod D_i, \prod D^{\mathrm{R}}_i, \prod \varphi_i)$, where $\prod \varphi_i(\prod d_i) = \prod \varphi_i(d_i)$, is a weak representation of $\prod X_i$.*

Proof. Clearly, $\prod \varphi_i$ is onto and continuous. □

It is possible to adapt a counterexample from [14] that shows that neither representations nor pseudo-open representations are uniformly closed under products. However, open representations and retract representations are uniformly closed under product.

Proposition 6.7 *Let $(D_i, D^{\mathrm{R}}_i, \varphi_i)$, for $i \in I$, be an open representation of X_i. Then $(\prod D_i, \prod D^{\mathrm{R}}_i, \prod \varphi_i)$ is an open representation of $\prod X_i$.*

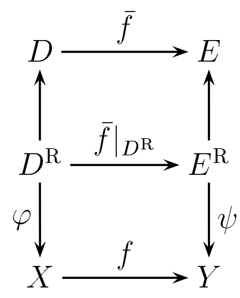

Fig. 3. Representing a function f.

Proof. By Lemma 6.6 it is a weak representation. A subbase for $\prod D_i$ are the sets $U_i \times \prod_{j \neq i} D_j$, where U_i is open in D_i, for $i \in I$. The images $\varphi_i[U_i \cap D_i^{\mathrm{R}}] \times \prod_{j \neq i} \varphi_j[D_j^{\mathrm{R}}]$ constitutes a subbase of $\prod X_i$, hence the result. $\square$

Proposition 6.8 *Let $(D_i, D_i^{\mathrm{R}}, \varphi_i, \eta_i)$, for $i \in I$, be a retract representation of X_i. Then $(\prod D_i, \prod D_i^{\mathrm{R}}, \prod \varphi_i, \prod \eta_i)$ is a retract representation of $\prod X_i$.*

Proof. By Lemma 6.6 $\prod \varphi_i$ is continuous. Clearly, $\prod \varphi_i \circ \prod \eta_i = \mathrm{id}$. The set $U_i \times \prod_{j \neq i} D_i$, where U_i is open in D_i, is a subbasic open set in $\prod D_i$.

$$(\prod \eta_i)^{-1}[(U_i \cap D_i^{\mathrm{R}}) \times \prod_{j \neq i} D_j^{\mathrm{R}}] = \eta_i^{-1}[U_i \cap D_i^{\mathrm{R}}] \times \prod_{j \neq i} X_i.$$

The right-hand side is a subbasic open set in $\prod X_i$ since η_i is continuous. Thus, $\prod \eta_i$ is continuous. $\square$

7 Functions and function spaces

7.1 Representing continuous functions

We will in this section study when functions between represented spaces can be represented. We start with the definition of the notion.

Definition 7.1 Let $(D, D^{\mathrm{R}}, \varphi)$ and $(E, E^{\mathrm{R}}, \psi)$ be domain representations of X and Y respectively. A continuous function $f\colon X \to Y$ is *represented* by a continuous function $\bar{f}\colon D \to E$ if $\psi(\bar{f}(x)) = f(\varphi(x))$, for all $x \in D^{\mathrm{R}}$. See Figure 3.

We now give sufficient conditions for a continuous function between representing domains to induce a continuous function. We merely state the following easy but important result.

Proposition 7.2 *Let $(D, D^{\mathrm{R}}, \varphi)$ be a representation of X and $(E, E^{\mathrm{R}}, \psi)$ be a weak representation of Y. Let $\bar{f}\colon D \to E$ be continuous such that $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$ and assume $\bar{f}$ respects the equivalence relations induced by φ and ψ. Then $\bar{f}$ induces a unique continuous function $f\colon X \to Y$.*

The following theorem tells us that for a large class of domain representations it is possible to represent any continuous function. In particular, representations of functions exist if the spaces are represented by domains constructed as in Sections 4.2 and 4.3.

Theorem 7.3 *Let $(D, D^{\mathrm{R}}, \varphi)$ be a dense weak representation of X, and let $(E, E^{\mathrm{R}}, \psi, \eta)$ be a retract representation of Y. Then every continuous function $f: X \to Y$ is represented by some continuous function $\bar{f}: D \to E$.*

Proof. The construction of $\bar{f}$ is done in two steps. First, let $f' = \eta \circ f \circ \varphi$. By hypothesis, this is a continuous function from D^{R} to E^{R}, which induces f since $\psi \circ f' = \psi \circ \eta \circ f \circ \varphi = f \circ \varphi$. The function f' may be considered as a continuous function from D^{R} to E as well since E^{R} has the subspace topology induced from E.

Secondly, the function f' is extended to a function $\bar{f}: D_c \to E$ by $\bar{f}(a) = \sqcap f'[\uparrow a \cap D^{\mathrm{R}}]$. The infimum is well-defined since $\uparrow a \cap D^{\mathrm{R}}$ is non-empty by density of D^{R}, and non-empty infima exist in consistently complete cpos. Clearly, $\bar{f}$ is monotone and hence it has a unique extension to D.

We now show that $\bar{f}$ is indeed an extension of f', i.e., that $\bar{f}(d) = f'(d)$ for $d \in D^{\mathrm{R}}$. Let $a \in \mathrm{approx}(d)$, then $\bar{f}(a) \sqsubseteq f'(d)$ since $d \in \uparrow a \cap D^{\mathrm{R}}$. Conversely, if $b \in \mathrm{approx}(f'(d))$, then $d \in f'^{-1}[\uparrow b]$. Hence, there exists $a \in \mathrm{approx}(d)$ such that $f'[\uparrow a \cap D^{\mathrm{R}}] \subseteq \uparrow b$, and so, $b \sqsubseteq \bar{f}(a) \sqsubseteq f'(d)$. Thus, $f'(d) = \bigsqcup_{a \in \mathrm{approx}(d)} \bar{f}(a) = \bar{f}(d)$. □

Constructions, such as the one in the theorem above, have been studied earlier, see for example [13].

7.2 Function spaces

In this section we consider representations of function spaces built by the function space construction on domains. Compare the work done by di Gianantonio [15] on representations of functions and functionals over the reals.

Let $(D, D^{\mathrm{R}}, \varphi)$ and $(E, E^{\mathrm{R}}, \psi)$ be representations of the topological spaces X and Y respectively. Let us further assume that every continuous function $f: X \to Y$ is represented by a continuous function $\bar{f}: D \to E$. Then let

$$[D \to E]^{\mathrm{R}} = \{ f \in [D \to E] : f[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}} \text{ and } (\forall x, y \in D^{\mathrm{R}})(\varphi(x) = \varphi(y) \implies \psi(f(x)) = \psi(f(y))) \}.$$

That is, $[D \to E]^{\mathrm{R}}$ consists of the continuous functions from D to E inducing continuous functions from X to Y. Thus there is an epimorphism $\vartheta: [D \to E]^{\mathrm{R}} \to (X \to Y)$ so that $([D \to E], [D \to E]^{\mathrm{R}}, \vartheta)$ is a representation of $X \to Y$, the continuous functions from X to Y. This representation induces a topology τ on $X \to Y$, the quotient topology obtained from the Scott topology on $[D \to E]$. The question is now how this topology is related to other topologies on $X \to Y$ and what properties it has.

Definition 7.4 Let X and Y be topological spaces.

(i) The sets $W(x, U) = \{f : f(x) \in U\}$, for $x \in X$ and U an open subset of Y, form a subbase for the *pointwise* topology on the function space $X \to Y$.

(ii) The sets $W(K,U) = \{f : f[K] \subseteq U\}$, for K a compact set in X and U an open set in Y, form a subbase for the *compact-open* topology on the function space $X \to Y$.

(iii) A topology on the function space $X \to Y$ is *jointly continuous* if the evaluation function eval: $(X \to Y) \times X \to Y$ defined by $\mathrm{eval}(f,x) = f(x)$ is continuous.

From general topology (see, e.g., [16]) we know that if a topology is jointly continuous then it is *finer* (has more open sets) than the compact-open topology and that the compact-open topology is finer than the pointwise topology.

From domain theory we know that the topology on function spaces of domains is exactly the pointwise topology and that the topology on function spaces is jointly continuous, hence the pointwise and the compact-open topology coincide for the function space construction on domains.

Since the Scott topology on function spaces of domains is jointly continuous a natural question is how close the induced topology is to being jointly continuous. It can be proved that under natural conditions, the induced topology is finer than the compact-open topology.

Lemma 7.5 *Let $(D, D^{\mathrm{R}}, \varphi)$ be a retract representation, and let $(E, E^{\mathrm{R}}, \psi)$ be a representation of Y. Suppose that every continuous function $f: X \to Y$ can be lifted to a continuous function $\bar{f}: D \to E$. Then the topology τ on $X \to Y$ induced by the representation $([D \to E], [D \to E]^{\mathrm{R}}, \vartheta)$ is finer than the compact-open topology.*

We will now show that if X is a locally compact Hausdorff space and if the representations are of a certain kind, then the topology τ induced on $X \to Y$ by the representation of the function space will be the compact-open topology. Moreover, τ will be jointly continuous if X is also regular.

Theorem 7.6 *Let $(D, D^{\mathrm{R}}, \varphi, \eta_X)$ be a dense retract representation with the closed image property of a locally compact Hausdorff space X and let $(E, E^{\mathrm{R}}, \psi, \eta_Y)$ be a retract representation of a space Y. Then $([D \to E], [D \to E]^{\mathrm{R}}, \vartheta, \epsilon)$ is a retract representation of $X \to Y$ with the compact-open topology.*

Proof. The embedding ϵ is obtained by a slight modification of the construction in Theorem 7.3. Let $f' = \eta_Y \circ f \circ \varphi$ and define $f'': D'_c \to E$ by $f''(a) = \sqcap f'[\uparrow a \cap D^{\mathrm{R}}]$, where D'_c is the set of all $a \in D_c$ such that $\varphi[\uparrow a \cap D^{\mathrm{R}}]$ is not only closed, but compact. Finally, let $\epsilon(f)(d) = \bigsqcup\{f''(a) : a \in \downarrow d \cap D'_c\}$.

The closed image property implies that if $b \in D_c$ is above some $a \in D'_c$, then $\varphi[\uparrow b \cap D^{\mathrm{R}}]$ is compact, i.e., $b \in D'_c$, since closed subsets of compact sets are compact. Thus, $\epsilon(f)(d)$ is the supremum of either an empty set or a directed set, i.e., ϵ is well-defined.

An argument similar to the one in Theorem 7.3 shows that $\epsilon(f)$ induces

f, i.e., $\vartheta \circ \epsilon = \mathrm{id}$. The argument requires that $\downarrow d \cap D'_c$ is non-empty. This follows from local compactness of X.

The induced topology τ is finer than the compact-open topology by Lemma 7.5. This implies that ϑ is continuous.

For continuity of ϵ, let $\epsilon(f) \in \uparrow\langle a; b\rangle$ with $a \in D_c$ and $b \in E_c$. By definition of ϵ we may restrict to $a \in D'_c$. Now,

$$\begin{aligned}
\epsilon(f) \in \uparrow\langle a; b\rangle &\iff b \sqsubseteq f''(a) \\
&\iff f'[\uparrow a \cap D^{\mathrm{R}}] \subseteq \uparrow b \\
&\iff f[\varphi[\uparrow a \cap D^{\mathrm{R}}]] \subseteq \eta_Y^{-1}[\uparrow b] \\
&\iff f \in W(\varphi[\uparrow a \cap D^{\mathrm{R}}], \eta_Y^{-1}[\uparrow b]).
\end{aligned}$$

□

The theorem is a generalisation of a result by di Gianantonio [15]. Note that this result cannot be lifted to functionals since function spaces fails to be locally compact in general.

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