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Effective domain representations of $\mathscr{H}(X)$, the space of compact subsets

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Abstract

This paper gives effective domain representations of spaces $\mathscr{H}(X)$ of non-empty compact subsets of effective complete metric spaces X. The domain representation of $\mathscr{H}(X)$ is constructed from a domain representation of X using the Plotkin power domain construction. As an application of the representation an effective version of a fundamental theorem on IFS (iterated function system) is shown. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we construct effective domain representations of $\mathscr{H}(X)$, the space of non-empty compact subsets of a complete effective metric space X. A domain representation of a topological space consists of a quotient map from a subset of a domain onto the topological space. The representation can be used to introduce computability on the topological space via the representing function. Since domain theory is a theory of approximations this is in essence a method to introduce effectivity on a space by computing on approximations of points in the space.

The representations of $\mathscr{H}(X)$ are constructed from a standard domain representation D of X using the Plotkin power domain construction. This construction gives us further evidence of the generality of using domain representations to introduce computability on uncountable structures [18, 4, 5].

Having a domain representation of the compact subsets of an effective metric space we can introduce a notion of *effectively compact* sets. We also introduce a notion of *effectively totally bounded* sets directly on the effective metric space. These two notions are proven to be equivalent. The notion of effectively totally bounded corresponds to the notion of compactness used in constructive mathematics.

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Using the effective domain representation of $\mathscr{H}(X)$ we can show an effective version of a fundamental theorem from the theory of IFS (iterated function system). We show that any finite set of effective contractive functions has an attractor and that this attractor is effectively compact.

Edalat and Heckmann [10] have used continuous (non-algebraic) domains to build domain representations of $\mathscr{H}(X)$. See also [8, 9]. The approach presented here differs from theirs by using algebraic domains and by taking effectivity into account.

In Section 2 we present some basics of domain theory including the Plotkin power domain construction. In Section 3 we present the notion of domain representations. We include in this section also the construction of domain representations for metric spaces from [4] for reference.

Section 4 contains two constructions of domain representations of $\mathscr{H}(X)$. The first and simpler construction is for the case when effectivity is not considered. The construction is then repeated for the effective case. This construction is harder since consistency is not decidable in the naïve approach. This forces us to work with formal balls and results in an extra level of complexity.

2. Domains

In this section we will briefly review domain theory. We include a presentation of the Plotkin power domain construction. We concentrate on giving the notions and hint at some results. The proofs are generally omitted and can be found in [17] or in [1].

2.1. Preliminaries on domains

Let $D=(D, \sqsubseteq)$ be a partial order. Let $A\subseteq D$. We will use the notation $\uparrow A$ to denote the set $\{y\in D\colon \exists x\in A(x\sqsubseteq y)\}$. We will abbreviate $\uparrow\{x\}$ by $\uparrow x$. We define $\downarrow A$ and $\downarrow x$ dually. A set $A\subseteq D$ is *directed* if $A\neq\emptyset$ and whenever $x, y\in A$ then there is $z\in A$ such that $x\sqsubseteq z$ and $y\sqsubseteq z$. The supremum, or least upper bound, of a set $A\subseteq D$ (if it exists) is denoted by $\bigsqcup A$. As usual we write $x\sqcup y$ instead of $\bigsqcup\{x, y\}$.

A *complete partial order*, abbreviated *cpo*, is a partial order, $D=(D; \sqsubseteq, \perp)$, such that $\perp$ is the least element in D and where any directed set $A\subseteq D$ has a supremum, $\bigsqcup A$.

Let D be a cpo. Then an element $a\in D$ is *compact* if whenever $A\subseteq D$ is a directed set and $a\sqsubseteq\bigsqcup A$, then $a\in\downarrow A$. The set of compact elements of D is denoted by D_{c}.

A cpo D is *algebraic* if for each $x\in D$, the set $\mathrm{approx}(x)=\downarrow x\cap D_{\mathrm{c}}$ is directed and $x=\bigsqcup\mathrm{approx(x)}$. A cpo D is *consistently complete* if $\bigsqcup A$ exists in D whenever $A\subseteq D$ is a consistent set, i.e., has an upper bound.

Definition 2.1. A *Scott–Ershov domain* is a consistently complete algebraic cpo.

The unqualified word *domain* will usually stand for Scott–Ershov domain. However, since the Plotkin power domain of a Scott–Ershov domain need not be a Scott–Ershov domain we sometimes let *domain* stand for algebraic cpo. We trust to the reader to make the correct interpretation.

The topology normally used on domains is called the Scott topology. Let D be an algebraic cpo. A subset U of D is open if
(i) $x \in U$ and $x \sqsubseteq y$ implies $y \in U$, and
(ii) $x \in U$ implies that there exists $a \in \mathrm{approx}(x)$ such that $a \in U$.

An easy observation is that the Scott topology on a domain is T_0. However, the Scott topology fails to be T_1 on all domains except the trivial domain consisting of a single element.

One can show that the sets $\uparrow a$ for $a \in D_c$ constitutes a base for the Scott topology on D. We will also write B_a for $\uparrow a$.

Let D and E be algebraic cpos. A function $f: D \to E$ is Scott continuous if f is monotone and

$$f(\bigsqcup A) = \bigsqcup f[A],$$

for any directed $A \subseteq D$. The notion of Scott continuity coincides with the notion of continuity induced from the Scott topology on the algebraic cpos.

Any continuous function between domains is determined by its values on the compact elements. In fact, let D and E be domains, then a monotone function $f: D_c \to E$ has a unique extension to a continuous function $g: D \to E$ such that $f = g|_{D_c}$.

Domains are often constructed as the completion of some underlying structure. We will study the structure from which Scott–Ershov domains can be constructed.

The compact elements D_c of a Scott–Ershov domain D form a conditional upper semilattice with least element, abbreviated *cusl*. That is, a cusl is a partially ordered set, with a least element, where a least upper bound exists for every pair of elements that have an upper bound.

Definition 2.2. Let P be a preorder. Then $I \subseteq P$ is an *ideal* if
(i) $I \neq \emptyset$,
(ii) if $a \in I$ and $b \sqsubseteq a$ then $b \in I$, and
(iii) if $a, b \in I$ then there exists an upper bound of a and b in I.

Note that, when P is a cusl, then (iii) and (ii) in the definition above implies that if $a, b \in I$ then $a \sqcup b \in I$.

For $a \in P$ we let $[a]$ denote the principal ideal generated by a. The ideal completion over a cusl P is the set of all ideals over P, denoted $\mathrm{Idl}(P)$. When ordered by set inclusion the ideal completion of a cusl forms a Scott–Ershov domain. The compact elements of $\mathrm{Idl}(P)$ are the principal ideals $[a]$, for $a \in P$.

The representation theorem for Scott–Ershov domains tells us that any Scott–Ershov domain is the ideal completion of a cusl.

Theorem 2.3. *Let D be a Scott–Ershov domain. Then* $\mathrm{Idl}(D_c) \cong D$.

We clearly have the following equivalence, for $I \in \mathrm{Idl}(P)$

$$[a] \subseteq I \Leftrightarrow a \in I.$$

Thus the basic open sets of $\mathrm{Idl}(P)$ in the Scott topology are of the form $B_a = \{I \in \mathrm{Idl}(P)\colon a \in I\}$ for $a \in P$.

2.2. *Effective domains*

We start by recalling some basic notions of computability.

A *structure* A is a tuple $A = (A; R_1, \ldots, R_p; \sigma_1, \ldots, \sigma_q)$, where A is a nonempty set, $R_j \subseteq A^{n_j}$ is an n_j-ary relation and $\sigma_i : A^{m_i} \to A$ is an m_i-ary operation on A. A *numbering* of a structure A is a surjective function $\alpha : \Omega_\alpha \to A$, where $\Omega_\alpha \subseteq \omega$. Let $\equiv_\alpha$ denote the equivalence relation defined on Ω_α by

$$m \equiv_\alpha n \Leftrightarrow \alpha(m) = \alpha(n).$$

The pair (A, α), where α is a numbering of the structure A, is an *effective structure* if every function and relation can be tracked with respect to the numbering α. An effective structure (A, α) is *computable* if Ω_α is a recursive set and $\equiv_\alpha$ is a recursive relation.

Let (A, α) and (B, β) be effective structures. A function $f : A \to B$ is (α, β)-*computable* if there is a partial recursive function $\hat{f}$ such that $\Omega_\alpha \subseteq \mathrm{dom}\, \hat{f}$ and for each $m \in \Omega_\alpha$, $f(\alpha(m)) = \beta(\hat{f}(m))$.

A partial function $g : A \to B$ is (α, β)-*computable* if there is a partial recursive function $\hat{g}$ such that $\mathrm{dom}(g \circ \alpha) \subseteq \mathrm{dom}\, \hat{g}$, and which satisfies $g(\alpha(m)) = \beta(\hat{g}(m))$, for $m \in \mathrm{dom}(g \circ \alpha)$.

A set $C \subseteq A$ is α-*decidable* (α-*semidecidable*) if $\alpha^{-1}[C]$ is recursive (r.e.). A recursive (r.e.) index for $\alpha^{-1}[C]$ is called a recursive (r.e.) α-index of C.

When regarding computability on a cusl we are not only interested in having a decidable ordering but also having a decidable consistency relation and the ability to compute suprema of finite consistent sets. Therefore we consider a cusl to be a structure of the form $P = (P; \sqsubseteq, \mathrm{Cons}, \sqcup, \bot)$.

Definition 2.4. Let P be a cusl. Then (P, α) is a *computable cusl* if α is a computable numbering of the structure $P = (P; \sqsubseteq, \mathrm{Cons}, \sqcup, \bot)$. A domain D is an *effective domain* if there is α such that (D_c, α) is a computable cusl. We denote this effective domain by (D, α).

Definition 2.5. Let (D, α) be an effective domain. Then $x \in D$ is an α-*computable element* if $\mathrm{approx}(x)$ is α-semidecidable. An r.e. index of the set $\alpha^{-1}[\mathrm{approx}(x)]$ is an α-*index* of the computable element x.

The prefix α will be dropped when the numbering is clear from the context.

Let D_k denote the set $\{x \in D\colon x \text{ is computable}\}$. Note that $D_\mathrm{c} \subseteq D_k$.

Definition 2.6. Let (D,α) and (E,β) be effective domains. A continuous function $f: D \to E$ is (α,β)-*effective* if the relation $R \subseteq D_c \times E_c$ defined by

$$R(a,b) \Leftrightarrow b \sqsubseteq f(a)$$

is (α,β)-semidecidable, that is the relation

$$\hat{R}(m,n) \Leftrightarrow R(\alpha(m),\alpha(n))$$

is r.e. An r.e. index for $\hat{R}$ is an *effective index for* f with respect to α and β.

Lemma 2.7. *Let $(D,\alpha),(E,\beta)$ and (F,γ) be effective domains and let $f: D \to E$ and $g: E \to F$ be continuous and (α,β)-effective and (β,γ)-effective, respectively.*
(i) *If $x \in D$ is α-computable then $f(x) \in E$ is β-computable.*
(ii) *The composition $h = g \circ f$ is (α,γ)-effective.*

The proof is standard, see [17]. We observe that the standard proof is uniform. That is, we can uniformly obtain an index for $f(x)$ from indices for f and x. Similarly an index for h is obtained uniformly from indices of f and g.

Definition 2.8. Let (D,α) be a effective domain and suppose $D_c \subseteq C \subseteq D$. Then (C,γ) is a *constructive subdomain* of (D,α) if $\gamma: \Omega_\gamma \to C$ is a surjective numbering such that $\Omega_\gamma \subseteq \omega$ is recursive, and
(i) the inclusion mapping $\imath: D_c \to C$ is (α,γ)-computable, and
(ii) the relation $R(n,m) \Leftrightarrow \alpha(n) \sqsubseteq \gamma(m)$ is r.e., that is $\mathrm{approx}(\gamma(m))$ is α-semidecidable uniformly in m.

Note that γ-equality is not decidable in general.

We state the following theorem which can be found in [17].

Theorem 2.9. *Let (D,α) be a effective domain. Then there is a numbering $\bar{\alpha}: \omega \to D_k$ such that $(D_k,\bar{\alpha})$ is a constructive subdomain of (D,α).*

2.3. The Plotkin power domain

Power domains were introduced by Plotkin [16]. They are used to give semantics to programs that are non-deterministic or parallel. A power domain construction is a domain theoretic variant of the powerset operator.

Power domains are usually one of three kinds, the upper, the lower and the convex power domain, which is a combination of the other two. They are often associated with the names Smyth, Hoare and Plotkin, respectively.

The class of Scott–Ershov domains are closed under the Smyth and the Hoare power domain construction. However, it is not closed under the Plotkin power domain construction. We will therefore consider a generalisation of *countably based* Scott–Ershov domains, called *SFP domains*. The acronym SFP stands for *sequence of finite partial* *orders*. It is well known that the category of SFP domains is cartesian closed and closed under the Plotkin power domain.

We will use the notation $\mathscr{P}_f^*(X)$ to denote all finite non-empty subsets of the set X.

Definition 2.10. An *SFP domain* is a countably based algebraic cpo, D, which satisfies the following conditions. For any $A \in \mathscr{P}_f^*(D_c)$,
(i) $\mathrm{MUB}(A)$ is complete, that is, every upper bound of A is above some element of $\mathrm{MUB}(A)$, and
(ii) $\bigcup_{n\in\omega} \mathrm{MUB}^n(A)$ is finite,
where $\mathrm{MUB}(A)$ denotes the set of minimal upper bounds of A and $\mathrm{MUB}^n(A)$ is defined by

$$\mathrm{MUB}^0(A) = A,$$

$$\mathrm{MUB}^{n+1}(A) = \bigcup \{\mathrm{MUB}(B)\colon\ B \in \mathscr{P}_f(\mathrm{MUB}^n(A))\}.$$

Clearly, any countably based Scott–Ershov domain is an SFP domain.

We denote $\mathrm{MUB}(\{a,b\})$ by $\mathrm{mub(a,b)}$. Let K_n denote the finite set of numbers with index n under some canonical enumeration of all finite subsets of ω.

Definition 2.11. An SFP domain D is *effective* if there exists a computable numbering $\alpha : \omega \to D_c$ of $(D_c, \sqsubseteq, \bot)$ and a recursive function $f : \omega^2 \to \omega$ such that $\alpha[K_{f(n,m)}] = \mathrm{mub}(\alpha(n), \alpha(m))$.

The following lemma shows that the notion of effective SFP domain is a generalisation of the notion of effective Scott–Ershov domain.

Lemma 2.12. (i) *An effective Scott–Ershov domain is an effective SFP domain.*

(ii) *A Scott–Ershov domain that is an effective SFP domain is an effective Scott–Ershov domain.*

Proof. (i) is easy. For (ii) assume that D is a Scott–Ershov domain that is an effective SFP domain. Let $a, b \in D$. The supremum of a and b exists if $\mathrm{mub}(a,b) \neq \emptyset$. This is decidable and hence the consistency relation is decidable. To compute $a \sqcup b$ we simply have to compute the unique element in $\mathrm{mub}(a,b)$ if a and b are consistent. $\square$

Definition 2.13. Let D be an algebraic cpo and define a preorder $\sqsubseteq_P$ on $\mathscr{P}_f^*(D_c)$ by

$$A \sqsubseteq_P B \ \Leftrightarrow\ (\forall b \in B)(\exists a \in A)(a \sqsubseteq b), \text{ and} \tag{1}$$

$$(\forall a \in A)(\exists b \in B)(a \sqsubseteq b). \tag{2}$$

This is the *Egli–Milner* order. The conditions 1 and 2 will be referred to as the *Smyth* and *Hoare condition*, respectively. Note that the Egli–Milner order is not antisymmetric. The *Plotkin power domain* of D, $\mathscr{P}_P(D)$, is the ideal completion $\mathrm{Idl}(\mathscr{P}_f^*(D_c), \sqsubseteq_P)$.

The Smyth and the Hoare power domains are constructed similarly, but where the preorder is defined by using only the Smyth condition and only the Hoare condition, respectively.

The compact elements of $\mathscr{P}_P(D)$ are the ideals on the form $[A]=\{B\in\mathscr{P}_f^*(D_c\colon B\sqsubseteq_P A\}$, for $A\in\mathscr{P}_f^*(D_c)$. Note that $[A]=[B]$ does not imply that $A=B$ since $\sqsubseteq_P$ fails to be antisymmetric.

Lemma 2.14. *Let S be a non-empty subset of D. Then the set*

$$I=\left\{A\subseteq_f \bigcup_{x\in S}\mathrm{approx}(x)\colon \forall x\in S\,\exists a\in A(a\sqsubseteq x)\right\}$$

is an ideal in $\mathscr{P}_P(D)$.

Proof. Clearly $I\subseteq\mathscr{P}_f^*(D_c)$. The set I is non-empty since $\{\bot\}\in I$.

Let $B\sqsubseteq_P A$ for some $A\in I$. For any $b\in B$ there exists an $a\in A$ such that $b\sqsubseteq a$ by the Hoare condition. This a is an approximation to some $x\in S$, hence b is also an approximation of x. For any $x\in S$ there exists an $a\in A$ such that $a\sqsubseteq x$. By the Smyth condition there exists a $b\in B$ such that $b\sqsubseteq a\sqsubseteq x$. Thus $B\in I$, i.e., I is closed downwards with respect to $\sqsubseteq_P$.

Let $A,B\in I$ and let $C=\{c\in\mathrm{mub}(a,b) : a\in A, b\in B, \exists x\in S(c\sqsubseteq x)\}$. Clearly, any element of C is below some $x\in S$. For any $x\in S$ there exists $a\in A$ and $b\in B$ below x. Hence $\exists c\in\mathrm{mub}(a,b)(c\sqsubseteq x)$. This c belongs to C so $C\in I$. Any element of C is clearly above some element of A. For an $a\in A$ there exists an $x\in S$ above a. There exists a $b\in B$ below x, hence a and b is below x. Thus there exists an $c\in\mathrm{mub}(a,b)$ which also belongs to C. Thereby showing that any element of A is below some element of C. Hence $A\sqsubseteq_P C$. By the same argument $B\sqsubseteq_P C$. Hence C is an upper bound of A and B in I.

Thus we have shown that I is an ideal in $\mathscr{P}_P(D)$. □

We denote the ideal I constructed from the set S in the lemma above by $[S]$. This extends the notation of compact elements introduced above.

Lemma 2.15. *Let S be a finite subset of D_k, the computable elements of D. Then $[S]\in(\mathscr{P}_P(D))_k$ uniformly.*

Proof. The ideal I constructed in Lemma 2.14 is clearly computable under the assumption. □

Lemma 2.16. *Let D be an SFP domain and let $E=\mathscr{P}_P(D)$. Suppose that $[C]$ is a minimal upper bound of $[A]$ and $[B]$ in E_c. Then there exists a $C'\subseteq_f D_c$ such that $[C]=[C']$ and such that*

$$C'\subseteq\bigcup_{\substack{a\in A\\ b\in B}}\mathrm{mub}(a,b).$$

Proof. For any $c \in C$ there exists $a \in A$ and $b \in B$ such that $a, b \sqsubseteq c$ by the Smyth condition, hence c is an upper bound of some $a \in A$ and $b \in B$. If c is an upper bound of $a \in A$ and $b \in B$ then $d_{a,b,c}$ denotes a minimal upper bound of a and b below c.

Let $C' = \{d_{a,b,c} : a \in A, b \in B, c \in C \text{ and } a, b \sqsubseteq c\}$. Clearly C' consists only of minimal upper bounds. It is easy to verify $[A], [B] \sqsubseteq [C'] \sqsubseteq [C]$. Hence, $[C] = [C']$, since $[C]$ is a minimal upper bound. □

We will now consider taking a power domain of an effective SFP domain. So suppose that (D, α) is an effective SFP domain with a numbering α and let $E = \mathscr{P}_P(D)$. Define $\beta : \omega \to \mathscr{P}_f^*(D_c)$ to be the canonical numbering of all finite non-empty subsets of D_c. Clearly, β can also be viewed as a numbering of E_c.

Theorem 2.17. *Let (D, α) be an effective SFP domain and let $E = \mathscr{P}_P(D)$. Then (E, β) is an effective SFP domain, where β is as above.*

Proof. We will suppress the numberings in the argument below.

The order is clearly decidable since it is only finite quantifications over a decidable relation.

Let $[A]$ and $[B]$ be elements of E_c. If there exists an upper bound of $[A]$ and $[B]$ then there exists a minimal upper bound and by Lemma 2.16 there exists a minimal upper bound $[C]$ of $[A]$ and $[B]$ such that C only contains minimal upper bounds. Hence, if $[A]$ and $[B]$ are consistent then there exists a minimal upper bound of $[A]$ and $[B]$ consisting of a non-empty subset of the set

$$S = \bigcup \{\mathrm{mub}(a, b) : \ a \in A, \ b \in B, \ \mathrm{Cons}(\{\mathrm{a}, \mathrm{b}\})\}.$$

The set S is finite and we can generate all non-empty subsets of it effectively. Since the order is decidable we can determine which subsets are actually upper bounds. Let

$$\mathscr{C} = \{C \subseteq S : \ [C] \text{ is an upper bound of } [A] \text{ and } [B]\}.$$

If $\mathscr{C} \neq \emptyset$ then $[A]$ and $[B]$ are consistent. To find all the minimal upper bounds simply use the decidability of the order to find the minimal elements in $\mathscr{C}$. This is a complete set of minimal upper bounds. □

3. Domain representations

We start by giving the fundamental definition of domain representability.

Definition 3.1. Let X be a topological space and D^{R} be a subset of a domain D. Then $(D, D^{\mathrm{R}}, \varphi)$ is a *domain representation* of X if $\varphi : D^{\mathrm{R}} \to X$ is an onto quotient map. Suppose α is a numbering of D_c such that (D, α) is an effective domain. Then $((D, \alpha), D^{\mathrm{R}}, \varphi)$ is an *effective domain representation* of X.

The set D^{R} consists of the *representing* elements, hence the superscript R.

Since the function φ in the definition above is a quotient map it is the case that

$$D^{\mathrm{R}}/\!\sim\,\cong X,$$

where $\sim$ is the equivalence relation induced by φ on D^{R}, i.e., $x \sim y \Leftrightarrow \varphi(x)=\varphi(y)$ and where $D^{\mathrm{R}}/\!\sim$ is given the quotient topology from the Scott topology on D.

The representing domain D contains both proper approximations and total or complete representations of elements of X, the latter constituting the set D^{R}.

Our domain representations are usually *pseudo-open* domain representations, i.e., the representing function φ is actually pseudo-open. A function $f: X \to Y$ is *pseudo-open* if for any $y \in Y$ and any open set $U \subseteq X$ it is the case that

$$f^{-1}[y] \subseteq U \Rightarrow y \in (f[U])^{\circ},$$

where Z° denotes the interior of Z. This notion is due to Arhangelskij [2]. A continuous pseudo-open function is always a quotient.

Definition 3.2. Let $(D, D^{\mathrm{R}}, \varphi)$ and $(E, E^{\mathrm{R}}, \psi)$ be domain representations of X and Y, respectively. A continuous function $f: X \to Y$ is *represented* by a continuous function $\bar{f}: D \to E$ if $\psi(\bar{f}(x)) = f(\varphi(x))$, for all $x \in D^{\mathrm{R}}$.

The following easy result is important.

Proposition 3.3. *Let $(D, D^{\mathrm{R}}, \varphi)$ and $(E, E^{\mathrm{R}}, \psi)$ be domain representations of X and Y, respectively. Let $\bar{f}: D \to E$ be continuous such that $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$ and such that $\varphi(x)=\varphi(y) \Rightarrow \psi(\bar{f}(x)) = \psi(\bar{f}(y))$, for $x, y \in D^{\mathrm{R}}$. Then $\bar{f}$ induces a unique continuous function $f: X \to Y$.*

3.1. Domain representations of metric spaces

We will repeat the construction of domain representations of metric spaces from [4] and at the same time introduce some notions and notations for later use. The representations constructed here will be referred to as *standard representations*. A more detailed account of the constructions is given in [4].

Let X be a metric space. Our first aim is to build a domain representation $(D, D^{\mathrm{R}}, \varphi)$ of X. For a metric space X, we denote the interior and closure of a subset F of X by F° and $\overline{F}$, respectively. We denote the open sphere centred in $x \in X$ and with radius r by $B(x, r)$.

Definition 3.4. Let X be a metric space and let P be a family of nonempty closed subsets of X including X. Then $P = (P; \sqsubseteq, X)$ is a *closed neighbourhood system* if the following is satisfied:

(i) if $F, F' \in P$ and $F \cap F' \neq \emptyset$ then $F \cap F' \in P$, and
(ii) if $x \in U$, where U is open, then $(\exists F \in P)(x \in F^{\circ} \wedge F \subseteq U)$.

Condition (i) makes P ordered by reverse inclusion into a cusl. Thus the supremum $F \sqcup F'$ in P is $F \cap F'$. P is an approximation for X in the sense of [18]. Every metric space, X, has a closed neighbourhood system, since if $x \in U$ and U is open, then by regularity of X, there exists an F closed in X that fulfills (ii). Thus the set of all nonempty closed sets is a closed neighbourhood system.

Fix $X = (X, d)$ to be a complete metric space and fix a closed neighbourhood system P for X. Let $D = \mathrm{Idl}(P)$, the ideal completion of P. By the representation theorem for domains we know that $D_c \cong P$. We will often identify elements of D_c with the corresponding elements in P. For $I \in D$ we say that x is approximated by I if $x \in \bigcap I$.

For a subset $F \subseteq X$ let

$$\mathrm{diam}(F) = \sup_{x,y \in F} d(x, y).$$

Definition 3.5. An ideal $I \in D$ is a *converging ideal* if

$$(\forall \varepsilon > 0)(\exists F \in I)(\mathrm{diam}(F) < \varepsilon).$$

It is easy to see that the intersection of a converging ideal is a singleton set. We denote by $I \to x$ that I is a converging ideal such that $\bigcap I = \{x\}$. Let D^{R} be the set of converging ideals and define $\varphi : D^{\mathrm{R}} \to X$ by

$$\varphi(I) = x \Leftrightarrow I \to x.$$

Using the construction above the following result is shown in [4]. The result can actually be strengthened to include non-complete metric spaces.

Theorem 3.6. *Let X be a complete metric space. Then there exists a domain representation $(D, D^{\mathrm{R}}, \varphi)$ of X.*

A domain representation of a metric space constructed as above will be called a *standard* domain representations of the metric space. It can be shown [6] that standard domain representations of metric spaces are pseudo-open. Moreover, if $(D, D^{\mathrm{R}}, \varphi)$ and $(E, E^{\mathrm{R}}, \psi)$ are standard domain representations of the metric spaces X and Y, respectively, then it can be shown that any continuous function $f : X \to Y$ can be represented by a continuous function $\bar{f} : D \to E$.

The domain representation constructed above is not always effective. We will identify a subclass of metric spaces for which a general construction of effective domain representations is possible.

First we formalise the notion of an effective metric space. An effective metric space is a subspace of the metric completion of a computable metric space. A computable metric space is a computable set together with a computable metric on the set with values in a computable ordered archimedean field. Here are the precise definitions.

Definition 3.7. Let K be an ordered field. Then (K, γ) is a *computable ordered field* if γ is a computable numbering of the structure $K = (K; \leqslant, +, \cdot, 0, 1)$.

We let $\mathbb{R}_k$ denote the structure of recursive reals with its canonical effective numbering ρ. Let (K,γ) be a computable archimedean ordered field. Then K is isomorphic to a subfield of $\mathbb{R}$, see for example [7]. Furthermore [14] proves that any computable ordered subfield of $\mathbb{R}$ is computably embedded into $(\mathbb{R}_k,\rho)$. That is, there is a (γ,ρ)-computable embedding of K into $\mathbb{R}_k$.

Definition 3.8. A *computable metric space* is a triple $((A,\alpha),(K,\gamma),d)$, where
(i) (A,α) is a computable set,
(ii) (K,γ) is a computable archimedean ordered field,
(iii) $d: A^2 \to K$ is an (α,γ)-computable metric on A, that is, d is a metric and there exists a recursive function $\hat{d}: \Omega_\alpha^2 \to \Omega_\gamma$ satisfying $\gamma \circ \hat{d} = d \circ \alpha^2$, i.e., $\hat{d}$ tracks d.

We will abbreviate the notation of a computable metric space $((A,\alpha),(K,\gamma),d)$ by (A,K,d) or (A,α) when appropriate.

Given a metric space (A,d) we let A^* denote the metric completion of A with respect to the metric d. We let d denote the induced metric on A^* as well.

Definition 3.9. Let $((A,\alpha),(K,\gamma),d)$ be a computable metric space and let X satisfy $A \subseteq X \subseteq A^*$. Then $X = (X,(A,\alpha),(K,\gamma),d)$ is an *effective metric space.*

In this paper, we are mainly concerned with the case $X = A^*$.

A first example of a computable metric space is $\mathbb{Q}$ together with the canonical computable numbering of $\mathbb{Q}$, where the distance function takes values in the computable ordered field $\mathbb{Q}$ and is defined by $d(x,y) = |x-y|$. Note that the metric completion of $\mathbb{Q}$ is $\mathbb{R}$, so $\mathbb{R}$ is an effective metric space.

It is shown in [4] that $\mathbb{R}^n$, $C[0,1]$ (the continuous functions on the unit interval) and some L^p-spaces are effective metric spaces.

We now set out to represent effective metric spaces by effective domains.

Below we let $((A,\alpha),(K,\gamma),d)$ be a computable metric space. We will show that we can represent the effective metric space $A^* = (A^*,(A,\alpha),(K,\gamma),d)$ by an effective domain. The domain will consist of approximations of elements of A. A straightforward choice is to let the approximations be certain closed subsets of A ordered by reverse inclusion. However, it is often impossible to compute on such approximations. For example, to decide whether two approximations are consistent, i.e., have a nonempty intersection, we intuitively have to enumerate elements of A until we find an element that belongs to both of the approximations, which clearly is only a semidecidable process in general.

As approximations we are going to consider all closed balls with rational radii centred in points belonging to A. We will define operations on these balls considered as formal objects and not as sets, and thus the balls will be called *formal balls*, and denoted by $F_{a,r}$, where $a \in A$ and $r \in \mathbb{Q}_+$. We will sometimes use the set theoretic notations $x \in F_{a,r}$ and $F_{a,r} \subseteq V$ when $F_{a,r}$ is considered as the set $\{x \in A^*\colon\ d(a,x) \leqslant r\}$.

It will help to think about the following concepts when formal balls are interpreted as closed spheres in, e.g., $\mathbb{R}^2$.

Two formal balls $F_{a,r}, F_{b,s}$ are *consistent*, denoted $F_{a,r} \uparrow F_{b,s}$, if $d(a,b) \leqslant r+s$. Note that two formal balls may be consistent even though their intersection is empty. A finite set of formal balls is *pairwise consistent* if every pair of formal balls are consistent.

A formal ball $F_{a,r}$ is *contained* in another formal ball $F_{b,s}$, denoted $F_{a,r} \preccurlyeq F_{b,s}$, if $d(a,b)+r \leqslant s$. This containment relation implies set theoretic inclusion, and is clearly transitive. A finite set of formal balls is *permissible* if it is pairwise consistent and has no formal ball contained in another, i.e.,

$$\{F_{a_1,r_1}, \ldots, F_{a_n,r_n}\} \text{ permissible} \Leftrightarrow$$
$$\forall i,j (1 \leqslant i < j \leqslant n \Rightarrow F_{a_i,r_i} \uparrow F_{a_j,r_j} \wedge F_{a_i,r_i} \not\preccurlyeq F_{a_j,r_j} \wedge F_{a_j,r_j} \not\preccurlyeq F_{a_i,r_i}).$$

Both consistency and containment are decidable because the metric function d takes values in the computable ordered field K. It follows that it is decidable whether or not a finite set of formal balls is permissible.

Let P be the set of all finite permissible sets of formal balls. Let σ and τ range over P. We are about to make P into a cusl. To do that we need to specify the ordering relation. We define for $m,n \geqslant 0$,

$$\{F_{a_i,r_i}\colon 1 \leqslant i \leqslant n\} \sqsubseteq \{F_{b_j,s_j}\colon 1 \leqslant j \leqslant m\} \Leftrightarrow \forall i < n \, \exists j < m (F_{a_i,r_i} \preccurlyeq F_{b_j,s_j}).$$

This is a partial order because the relation is defined on permissible sets of formal balls. The empty set is the least element. We note that two permissible sets $\sigma = \{F_{a_i,r_i}\colon 1 \leqslant i \leqslant n\}$ and $\tau = \{F_{b_j,s_j}\colon 1 \leqslant j \leqslant m\}$ are consistent if for every i and j, F_{a_i,r_i} and F_{b_j,s_j} are consistent. The supremum of σ and τ is $\sigma \sqcup \tau = g(\sigma \cup \tau)$, whenever σ and τ are consistent, where the function g takes a consistent set of formal balls into a permissible set of formal balls by removing any formal ball containing another formal ball. We have shown that $(P; \sqsubseteq, \perp)$ is a cusl.

The set P ordered as above is a computable cusl. Let D be the effective domain $\mathrm{Idl}(P)$.

Definition 3.10. (i) An element $x \in A^*$ is *approximated* by $I \in D$ if

$$(\forall \sigma \in I)(\forall F_{a,r} \in \sigma)(x \in F_{a,r}).$$

(ii) The ideal $I \in D$ is *converging* if for any $\varepsilon > 0$ there exists a formal ball $F_{a,r} \in I$ such that $r < \varepsilon$.

Every converging ideal approximates exactly one element in A^*. We will use the notation $I \to x$ to say that I is converging and that the unique element approximated by I is x. For each $x \in A^*$ let I_x be the ideal generated by the set $\{F_{a,r}\colon d(a,x) < r \wedge r \in \mathbb{Q}_+ \wedge a \in A\}$. The ideal I_x converges to x and is the least ideal that converges to x.

Let D^{R} be the set of converging ideals in D. For every ideal $I \in D^{\mathrm{R}}$ there is an $x \in A^*$ such that $I \to x$, and conversely, for every $x \in A^*$ there is an ideal $I \in D^{\mathrm{R}}$ such that $I \to x$.

Define $\varphi: D^{\mathrm{R}} \to X$ by

$$\varphi(I) = x \Leftrightarrow I \to x.$$

That φ is onto is witnessed by the ideals I_x.

We have the following result [4].

Theorem 3.11. *Let* $X = (X, (A, \alpha), (K, \gamma), d)$ *be an effective metric space. Then* $(D, D^{\mathrm{R}}, \varphi)$ *constructed as above is an effective domain representation of* X.

We call the representation constructed above a *standard effective domain representation* of an effective metric space.

4. Representing compact subsets

In this section we will show that there exists a natural domain representation of the set of non-empty compact subsets of a complete metric space using the Plotkin power domain. The representation constructed here uses algebraic domains as opposed to the work of Edalat and Heckmann [10] which uses continuous domains.

Definition 4.1. Let X be a complete metric space. Then

$$\mathscr{H}(X) = \{K \subseteq X \colon K \neq \emptyset \text{ and } K \text{ is compact}\}.$$

4.1. Representing compact subsets of a complete metric space

We start by giving the construction for the easier case when effectivity is not considered. The effective case will be studied in the next subsection.

Let $(D, D^{\mathrm{R}}, \psi)$ be the domain representation constructed as in Theorem 3.6 of the complete metric space X and let $E = \mathscr{P}_P(D)$. Recall that the compact elements of D are the principal ideals generated over a closed neighbourhood system, cf. Definition 3.4.

Definition 4.2. Define a function diam: $\mathscr{P}_f^*(D_{\mathrm{c}}) \to [0, \infty) \cup \{\infty\}$ by

$$\operatorname{diam}(A) = \max\{\operatorname{diam} S \colon S \in A\}.$$

Lemma 4.3. *For* A *and* B *in* $\mathscr{P}_f^*(D_{\mathrm{c}})$ *we have that* $A \sqsubseteq_P B \Rightarrow \operatorname{diam}(B) \leqslant \operatorname{diam}(A)$.

Proof.

$$\begin{aligned} A \sqsubseteq_P B &\Rightarrow (\forall T \in B)(\exists S \in A)(S \sqsubseteq T) \\ &\Leftrightarrow (\forall T \in B)(\exists S \in A)(T \subseteq S) \\ &\Rightarrow (\forall T \in B)(\exists S \in A)(\operatorname{diam}(T) \leqslant \operatorname{diam}(S)) \\ &\Rightarrow \operatorname{diam}(B) \leqslant \operatorname{diam}(A), \end{aligned}$$

where the first implication is the Smyth condition of the Egli–Milner order. □

The lemma above shows that diam is well defined on E_c and that it is monotonically decreasing.

Definition 4.4. Extend the function diam to a function diam: $E \to [0, \infty) \cup \{\infty\}$ by

$$\mathrm{diam}(I) = \inf_{A \in I} \mathrm{diam}(A).$$

An ideal I is *converging* if $\mathrm{diam}(I) = 0$. The set of converging ideals in E is denoted by E^{R}.

The function diam is a monotonically decreasing function, hence large ideals will give small values when diam is applied to them. The converging ideals are, as we will see, in some sense the ideals that contain total information about the represented object. We will introduce a natural representation function φ and will show that any converging ideal represents a non-empty compact subset of the space X.

Definition 4.5. Define a function $\varphi: \mathscr{P}_f^*(D_c) \to \mathscr{P}(X)$ by

$$\varphi(A) = \bigcup_{S \in A} S.$$

Lemma 4.6. *Let* $A, B \in \mathscr{P}_f^*(D_c)$. *Then* $A \sqsubseteq_P B \Rightarrow \varphi(B) \subseteq \varphi(A)$.

Proof.

$$\begin{aligned} A \sqsubseteq_P B &\Rightarrow (\forall T \in B)(\exists S \in A)(S \sqsubseteq T) \\ &\Leftrightarrow (\forall T \in B)(\exists S \in A)(T \subseteq S) \\ &\Rightarrow \varphi(B) \subseteq \varphi(A). \qquad \square \end{aligned}$$

The lemma above shows that φ is well-defined on E_c and that it is monotonically decreasing w.r.t. $\subseteq$.

Definition 4.7. Extend φ to a function $\varphi: E \to \mathscr{P}(X)$ by

$$\varphi(I) = \bigcap_{A \in I} \varphi(A).$$

The function φ takes E_c to closed sets since $\varphi(A)$ for $A \in \mathscr{P}_f^*(D_c)$ is a finite union of closed sets. Moreover φ takes E to closed sets since $\varphi(I)$ for $I \in E$ is an intersection of closed sets.

The following technical lemma is crucial. It says that no set appearing in a converging ideal will be avoided by the set represented by the ideal.

Lemma 4.8. *Let I be a converging ideal. Then for any $A \in I$ and $S \in A$ there exists an $x \in S \cap \varphi(I)$.*

Proof. Fix $A \in I$ and $S \in A$. Since I is a converging ideal we can find a sequence $(A_n)_{n\in\omega}$ such that $A = A_0$ and $A_n \sqsubseteq_P A_{n+1}$, for all n, and such that $\lim_n \operatorname{diam}(A_n) = 0$. Define inductively a sequence $(S_n)_{n\in\omega}$ by letting $S_0 = S \in A = A_0$ and by letting $S_{n+1} \in A_{n+1}$ be such that $S_n \sqsubseteq S_{n+1}$. This is possible by the Hoare condition. For each n select a point x_n in S_n. The sequence (x_n) is a Cauchy sequence since $\operatorname{diam} S_n \to 0$ as $n \to \infty$. Let $x = \lim_n x_n$. We will show that $x \in \varphi(I)$ by showing that $x \in \varphi(B)$ for an arbitrary $B \in I$. So let $B \in I$ and define inductively a sequence $(C_n)_{n\in\omega}$ in I by letting C_0 be such that $A_0 \sqsubseteq_P C_0$ and $B \sqsubseteq_P C_0$ and by letting C_{n+1} be such that $C_n \sqsubseteq_P C_{n+1}$ and $A_{n+1} \sqsubseteq_P C_{n+1}$. Define a sequence $(T_n)_{n\in\omega}$ by letting $T_0 \in C_0$ be such that $S_0 \sqsubseteq T_0$ and by letting $T_{n+1} \in C_{n+1}$ be such that $T_n \sqsubseteq T_{n+1}$ and $S_{n+1} \sqsubseteq T_{n+1}$. For each n select a point y_n in T_n. Let $y = \lim_n y_n$. For any $m > n$ we have that x_m and y_m is in S_n since $S_m \subseteq S_n$. Hence, since $\operatorname{diam} S_n \to 0$, $x = y$ and $x \in S_n$. Since $B \sqsubseteq_P C_0$, there exists by the Smyth condition an $R \in B$ such that $R \sqsubseteq T_0$. Hence $x \in \varphi(B)$. Since the choice of B was arbitrary we have $x \in \varphi(I)$. $\square$

We observe that the proof above uses both the Smyth and the Hoare condition on the ordering of E.

Proposition 4.9. *Let I be a converging ideal. Then $\varphi(I)$ is a non-empty compact subset of X.*

Proof. We have already noted that $\varphi(I)$ is closed. Hence it is sufficient to show that $\varphi(I)$ is non-empty and totally bounded, since a closed and totally bounded subset of a complete metric space is compact. That $\varphi(I)$ is non-empty follows by Lemma 4.8.

We now show that $\varphi(I)$ is totally bounded by showing that we can find an ε-net for any $\varepsilon > 0$. Let $\varepsilon > 0$. Choose an $A \in I$ such that $\operatorname{diam}(A) < \varepsilon$. By Lemma 4.8 there exists an $x_S \in S \cap \varphi(I)$ for every $S \in A$. Clearly $\{x_S : S \in A\}$ is an ε-net for $\varphi(I)$. $\square$

Lemma 4.10. *For any non-empty compact subset K of X there exists a converging ideal I such that $\varphi(I) = K$.*

Proof. Let

$$I = \{A \in \mathscr{P}_f^*(D_c)\colon\ \forall S \in A (S \cap K \neq \emptyset) \text{ and } K \subseteq \varphi(A)\}.$$

It is easy to verify that the set

$$C = \{S \cap T\colon\ S \in A,\ T \in B,\ S \cap T \cap K \neq \emptyset\},$$

is an upper bound of A, B. By definition we have that any set in C intersects K, hence to see that C belongs to I we only have to verify that $K \subseteq \varphi(C)$. This is the case since for any $x \in K$ there exists $S \in A$ and $T \in B$ such that $x \in S \cap T$. Thus, I contains upper bounds of any two sets in I. Let $B \sqsubseteq_P A$. Then any set in B intersects K since there exists a smaller set in A that intersects K by the Hoare condition. By Lemma 4.6

$K \subseteq \varphi(A) \subseteq \varphi(B)$. Hence I downwards closed. Thus I is an ideal since it is clearly non-empty. Since K is compact there exists $A \in I$ such that $\mathrm{diam}(A) < \varepsilon$ for any $\varepsilon > 0$. □

We will denote the ideal I in the proof of the lemma above by I^K.

Collecting the results in this section we have the following theorem.

Theorem 4.11. *Let $(D, D^{\mathrm{R}}, \psi)$ be a standard domain representation of a complete metric space X and let $E = \mathscr{P}_P(D)$. Let $\varphi : E^{\mathrm{R}} \to \mathscr{P}(X)$ be the function defined in Definition 4.7. Then $(E, E^{\mathrm{R}}, \varphi)$ is a domain representation of $\mathscr{H}(X)$.*

Note that the representing domain E in Theorem 4.11 is an algebraic cpo, but not necessarily a Scott–Ershov domain. If the space X is separable then we can choose a countably based standard representation D of X. Then clearly, D is an SFP domain and hence E is also an SFP domain.

Definition 3.1 required that the representing function φ in Theorem 4.11 should be a quotient. We have not considered topology on $\mathscr{H}(X)$ yet. However, if we consider $\mathscr{H}(X)$ with the quotient topology induced by φ, then clearly, φ is a quotient. It remains to determine what the induced topology is. We will now show that it is the topology induced by the Hausdorff metric on $\mathscr{H}(X)$.

Definition 4.12. Let X be a metric space. The Hausdorff metric, d_{H}, on $\mathscr{H}(X)$ is given by

$$d_{\mathrm{H}}(K, K') = \max\left(\sup_{x \in K} d(x, K'),\ \sup_{y \in K'} d(y, K)\right),$$

where $d(x, K) = \inf_{y \in K} d(x, y)$.

From now on we assume that $\mathscr{H}(X)$ is given the Hausdorff metric.

Let I and J be converging ideals containing A. Then for any $x \in \varphi(I)$ there exists an $S \in A$ such that $x \in S$. By Lemma 4.8 there exists an $x' \in S \cap \varphi(J)$. Hence $d(x, \varphi(J)) \leqslant \mathrm{diam}(S) \leqslant \mathrm{diam}(A)$. By symmetry $d_{\mathrm{H}}(\varphi(I), \varphi(J)) \leqslant \mathrm{diam}(A)$.

Lemma 4.13. *Let $(E, E^{\mathrm{R}}, \varphi)$ be the domain representation of $\mathscr{H}(X)$ in Theorem 4.11. Then $\varphi : E^{\mathrm{R}} \to \mathscr{H}(X)$ is continuous.*

Proof. It is sufficient to show that $\varphi^{-1}[B(K, r)]$ is open for an arbitrary basic open set $B(K, r)$. Let $I \in \varphi^{-1}[B(K, r)]$. Since I is converging there exists an $A \in I$ such that $\mathrm{diam}(A) < r - d_{\mathrm{H}}(K, \varphi(I))$. Let J be a converging ideal containing A. Then $d_{\mathrm{H}}(\varphi(I), \varphi(J)) \leqslant \mathrm{diam}(A) < r - d_{\mathrm{H}}(K, \varphi(I))$. Hence $d_{\mathrm{H}}(K, \varphi(J)) \leqslant d_{\mathrm{H}}(K, \varphi(I)) + d_{\mathrm{H}}(\varphi(I), \varphi(J)) < r$. Thus $I \in \uparrow A \cap E^{\mathrm{R}} \subseteq \varphi^{-1}[B(K, r)]$. □

The ideal I^K constructed in Lemma 4.10 contains every possible approximation of K. We would like to construct an ideal where the approximations are of a uniform and simple form. We will construct such an ideal, but in order to do that we will have to make a stipulation on the domain D representing X.

For the remaining part assume that the domain representation D of X is built from a closed neighbourhood system containing every closed sphere. For any $\varepsilon > 0$ the finite set of closed ε-spheres centred in an ε-net of K is a finite covering of K. If S is a closed sphere centred in x then let $\frac{1}{2}S$ denote the closed sphere centred in x with half the radius of that of S.

Let $K \in \mathscr{H}(X)$. Construct a sequence (A_n) of approximations of K by letting A_0 be the set of 2^1-spheres centred in a 2^0-net of K. Suppose A_n is constructed such that A_n consists of 2^{-n+1}-spheres centred in K such that $\{\frac{1}{2}S\colon S \in A_n\}$ covers K. For each $S \in A_n$ let A_S be the set of 2^{-n}-spheres centred in a 2^{-n-1}-net of $K \cap \frac{1}{2}S$. Let $A_{n+1} = \bigcup_{S \in A_n} A_S$. Clearly, A_{n+1} consists of 2^{-n} spheres centred in K. For any $x \in K$ there exists an $S \in A_n$ such that $x \in \frac{1}{2}S$. Since $\{\frac{1}{2}S'\colon S' \in A_S\}$ covers $K \cap \frac{1}{2}S$ there exists an $S' \in A_S$ such that $x \in \frac{1}{2}S'$. Hence $\{\frac{1}{2}S\colon S \in A_{n+1}\}$ covers K.

We will show that $A_n \sqsubseteq_P A_{n+1}$. Let $S' \in A_S$ for some $S \in A_n$. Then S' is a 2^{-n}-sphere centred within 2^{-n} from the centre of S. Hence S' is a subset of the 2^{-n+1}-sphere S. Thus the Smyth condition is verified. If $S \in A_n$ then there exists an $S' \in A_S \subseteq A_{n+1}$ such that $S' \subseteq S$ since $\{\frac{1}{2}S' : S' \in A_S\}$ covers $K \cap \frac{1}{2}S \neq \emptyset$.

Let

$$I_K = \bigcup_{n \in \omega} \downarrow A_n.$$

Then I_K is a converging ideal. However, observe that I_K is not uniquely determined by K since I_K depends on the ε-nets chosen in the construction.

Lemma 4.14. *Let $(D, D^{\mathrm{R}}, \psi)$ be a standard domain representation of X and suppose that the underlying closed neighbourhood system contains every closed sphere. Then $(E, E^{\mathrm{R}}, \varphi)$ is a pseudo-open domain representation of $\mathscr{H}(X)$.*

Proof. Let U be an open set such that $\varphi^{-1}[K] \subseteq U$. The ideal I_K belongs to U. There exists an $A \in I_K$ such that $\uparrow[A] \subseteq U$. In fact, we can choose A to be A_n, for some $n \in \omega$, in the construction of I_K.

Let $K' \in B(K, 2^{-n})$ and let $y \in K'$. Then there exists an $x \in K$ such that $d(x,y) < 2^{-n}$. Since $x \in K$ there exists an $S \in A_n$ such that $x \in \frac{1}{2}S$. But S is a 2^{-n+1}-sphere, hence any point within 2^{-n} from x will belong to S. In particular, $y \in S$. Thus $y \in \varphi(A_n)$, i.e., $K' \subseteq \varphi(A_n)$.

Let $S \in A_n$ and let x be the centre point of S. Since $d_{\mathrm{H}}(K, K') < 2^{-n}$ there exists a $y \in K'$ such that $d(x,y) < 2^{-n}$. Clearly, $y \in S$. Hence we have shown that $A_n \in I^{K'}$. It follows that $I^{K'} \in \uparrow[A_n] \subseteq U$ so $K' \in \varphi[U]$, i.e., $K \in (\varphi[U])^{\circ}$. □

Any pseudo-open map is a quotient map, hence the lemma above implies that φ is a quotient map.

Theorem 4.15. *Let $(D, D^{\mathrm{R}}, \psi)$ be a standard pseudo-open domain representation of a complete metric space X, where the underlying closed neighbourhood system contains* *every closed sphere of X. Then there exists a pseudo-open domain representation $(E, E^{\mathrm{R}}, \varphi)$ of the metric space $(\mathscr{H}(X), d_{\mathrm{H}})$.*

Proof. By Theorem 4.15 and Lemmas 4.13 and 4.14. □

It is not necessary to demand that every sphere is included in the closed neighbourhood system underlying the domain representation. A slight modification of the above proof shows that it suffices to have a set of spheres centred in a dense set of X and with rational radii in the underlying closed neighbourhood system.

4.2. Representing compact subsets of an effective metric space

This subsection will repeat the development in the previous subsection but taking effectivity into account. Analogous to the case for effective representations of metric spaces, the main difference in the development is that we will use formal balls with formal operations on them instead of working with subsets of the metric space.

The variable F, with or without subscripts, ranges over formal balls. The variables σ and τ range over permissible sets of formal balls. Many of the arguments use permissible sets of formal balls that are singleton sets, i.e., $\sigma = \{F\}$. We let $\mathbf{F}$ denote the singleton set $\{F\}$. Clearly, any $\mathbf{F}$ is a permissible set and $F \in \mathbf{F}$. The variables A, B and C range over finite sets of permissible sets of formal balls; we will identify these with the principal ideal that they generate, hence they also range over compact elements of the power domain. Finally, I and J range over the ideals in the power domain.

The following lemma shows that any compact set can be covered by finitely many formal balls centred in a dense subset of X.

Lemma 4.16. *For any compact set $K \subseteq X$ and any $\varepsilon > 0$ there exists a finite set M of formal balls such that* (*when considered as sets*)
(i) $K \subseteq \bigcup M$,
(ii) *each $F \in M$ has a radius less than or equal to ε, and*
(iii) $\forall F \in M (F \cap K \neq \emptyset)$.

Proof. Approximate an $\varepsilon/2$-net with points from the dense subset of X. Take the formal ε-balls centred in these points. □

From now on let $(D, D^{\mathrm{R}}, \psi)$ be the domain representation constructed in Theorem 3.11 of a complete *effective* metric space X and let $E = \mathscr{P}_P(D)$.

Intuitively the set represented by a set of formal balls is their intersection, but this intersection may be empty. The following definition avoids this by requiring that the sets are singleton sets.

Definition 4.17. An ideal $I \in E$ is *converging* if for any $\varepsilon > 0$ there exists $A \in I$ such that

$$A = \{\mathbf{F}_{a_1,r_1}, \ldots, \mathbf{F}_{a_n,r_n}\}$$

and such that $r_i < \varepsilon$ for $1 \leqslant i \leqslant n$. The set A is a set of sets of formal balls, but the latter sets contain only a single formal ball each. The set of converging ideals in E is denoted by E^{R}.

A converging ideal is, as we will see, in some sense an ideal that contains total information about the represented object. We will introduce a natural representation function φ and will show that the converging ideals represent the non-empty compact subsets of the space X.

Definition 4.18. Define a function $\varphi : D_{\mathrm{c}} \to \mathscr{P}(X)$ by

$$\varphi(\sigma) = \bigcap_{F_{a,r} \in \sigma} F_{a,r}$$

and extend this to $\varphi : \mathscr{P}_f^*(D_c) \to \mathscr{P}(X)$ by

$$\varphi(A) = \bigcup_{\sigma \in A} \varphi(\sigma).$$

The notation $F_{a,r}$ denotes both the formal ball and the set it contains. The interpretation should be clear from the context.

Note that the function φ has closed sets as values since they are finite unions and intersections of closed sets.

Lemma 4.19. *Let $A, B \in \mathscr{P}_f^*(D_{\mathrm{c}})$. Then $A \sqsubseteq_P B \Rightarrow \varphi(B) \subseteq \varphi(A)$.*

Proof.

$$\begin{aligned} A \sqsubseteq_P B &\Rightarrow (\forall \tau \in B)(\exists \sigma \in A)(\sigma \sqsubseteq \tau) \\ &\Leftrightarrow (\forall \tau \in B)(\exists \sigma \in A)(\forall F_{a,r} \in \sigma)(\exists F_{b,s} \in \tau)(F_{b,s} \preccurlyeq F_{a,r}) \\ &\Rightarrow (\forall \tau \in B)(\exists \sigma \in A)(\forall F_{a,r} \in \sigma)(\exists F_{b,s} \in \tau)(F_{b,s} \subseteq F_{a,r}) \\ &\Rightarrow (\forall \tau \in B)(\exists \sigma \in A)(\varphi(\tau) \subseteq \varphi(\sigma)) \\ &\Rightarrow \varphi(B) \subseteq \varphi(A). \qquad \Box \end{aligned}$$

The lemma above shows that φ is well defined as a function from E_{c} to $\mathscr{P}(X)$ and that it is monotonically decreasing.

Definition 4.20. Extend φ to a function $\varphi : E \to \mathscr{P}(X)$ by

$$\varphi(I) = \bigcap_{A \in I} \varphi(A).$$

The following lemma is the effective analog of Lemma 4.8. However, both its statement and proof are much more delicate. An ideal is *effective* if the ideal is a computable element of the effective domain, i.e., if the elements of the ideal can be enumerated effectively.

Lemma 4.21. *Let I be a converging ideal and let $A \in I$. If the singleton set $\mathbf{F}_{a,r}$ belongs to A, then for any $\varepsilon > 0$ there exists an $x \in X$ such that $x \in F_{a,r+\varepsilon} \cap \varphi(I)$. Moreover, such an x can be found effectively from ε and I when I is an effective ideal.*

Proof. Since I converges we can find a sequence $(A_n)_{n\in\omega} \in I$ such that A_n is of the form

$$A_n = \{\mathbf{F}_{a_{n1},r_{n1}}, \ldots, \mathbf{F}_{a_{nk_n},r_{nk_n}}\}$$

and such that $r_{ni} \leqslant 2^{-n}$, for all n and $i = 1, \ldots, k_n$.

We now show that if $\mathbf{F}_{b,s} \in B \in I$ then for any n there exists an $\mathbf{F}_{b',s'} \in A_n$ such that $F_{b,s}$ and $F_{b',s'}$ are consistent. Let $C \in I$ be an upper bound of A_n and B. Using the Hoare condition we can find $\sigma \in C$ such that $\mathbf{F}_{b,s} \sqsubseteq \sigma$ and using the Smyth condition there must exist an $\mathbf{F}_{b',s'} \in A_n$ such that $\mathbf{F}_{b',s'} \sqsubseteq \sigma$. We have that σ is an upper bound of $\mathbf{F}_{b,s}$ and $\mathbf{F}_{b',s'}$, hence consistency for the formal balls follows.

Choose an m such that $2^{-m+2} \leqslant \varepsilon$. For $k \in \omega$ we will choose a set containing a single formal ball in A_{m+k}. For $k = 0$ we choose $\mathbf{F}_{a_0,r_0} \in A_m$ such that F_{a_0,r_0} is consistent with $F_{a,r}$. Suppose $\mathbf{F}_{a_k,r_k}$ is chosen from A_{m+k}. Choose $\mathbf{F}_{a_{k+1},r_{k+1}} \in A_{m+k+1}$ such that $F_{a_{k+1},r_{k+1}}$ is consistent with F_{a_k,r_k}. Consider the sequence of centre points of these formal balls, i.e., the sequence (a_k). Clearly, this sequence is a Cauchy sequence in X, so let $x = \lim a_k$. The limit x belongs to $F_{a,r+\varepsilon}$ since

$$\begin{aligned} d(a,x) &\leqslant d(a,a_0) + \sum_{k=0}^{\infty} d(a_k, a_{k+1}) \\ &\leqslant r + r_0 + \sum_{k=0}^{\infty} (r_k + r_{k+1}) \\ &\leqslant r + 2^{-m} + \sum_{k=0}^{\infty} (2^{-m-k} + 2^{-m-k-1}) \\ &= r + 2\sum_{i=m}^{\infty} 2^{-i} \\ &= r + 2\cdot 2^{-m+1} \\ &\leqslant r + \varepsilon. \end{aligned}$$

We will now show that this x is in every $\varphi(B)$ for $B \in I$. So let $B \in I$. Since B is a finite set there must exist a $\sigma \in B$ such that σ and $\mathbf{F}_{a_k,r_k}$ are consistent for infinitely many k. Hence, for any $F_{b,s} \in \sigma$ the distance from b to x is bounded by $s + 2r_k \leqslant s + 2^{-k+1}$ for infinitely many k. Thus $x \in F_{b,s}$ and hence $x \in \varphi(B)$.

Assume that I is an effective ideal, i.e., is an r.e. set. To find A_n simply enumerate the members of I until one is found that satisfies the desired properties. It is decidable whether two formal balls are consistent, hence finding a consistent formal ball in A_n

is effective. The starting index m can be computed from ε. Thus, it is possible to effectively construct the sequence (a_k). This sequence converges effectively to x. $\square$

Proposition 4.22. *Let I be a converging ideal. Then $\varphi(I)$ is a non-empty compact subset of X.*

Proof. Clearly, $\varphi(I)$ is closed. Hence it is sufficient to show that $\varphi(I)$ is non-empty and totally bounded, since a closed and totally bounded subset of a complete metric space is compact. That $\varphi(I)$ is non-empty follows by Lemma 4.21.

We now show that $\varphi(I)$ is totally bounded by showing that we can find an ε-net for any $\varepsilon > 0$. Let $\varepsilon > 0$. Since I is converging there exists an $A \in I$ such that A is of the form

$$A = \{\mathbf{F}_{a_1,r_1}, \dots, \mathbf{F}_{a_k,r_k}\},$$

where r_i is bounded by $\varepsilon/3$. For each $i \in \{1, \dots, k\}$ there exists an

$$x_i \in F_{a_i, r_i + \frac{\varepsilon}{3}} \cap \varphi(I)$$

by Lemma 4.21. Clearly, the set of ε-balls centred in the x_i-s cover $\varphi(A) \supseteq \varphi(I)$, hence $\{x_i\colon 1 \leqslant i \leqslant k\}$ is an ε-net. $\square$

Corollary 4.23. *Let I be a converging ideal and let $A \in I$. If the singleton set $\mathbf{F}_{a,r}$ belongs to A, then there exists an $x \in X$ such that $x \in F_{a,r} \cap \varphi(I)$.*

Proof. Using Lemma 4.21 construct a sequence (x_n) of points in $\varphi(I)$, where $x_n \in F_{a,r+2^{-n}}$. Since $\varphi(I)$ is compact by Proposition 4.22 the sequence (x_n) has a cluster point $x \in \varphi(I)$. The point x belongs to $F_{a,r}$ since $F_{a,r}$ is closed and since $d(x, F_{a,r})$ is bounded by 2^{-n} for infinitely many n. $\square$

We will now show that for any compact subset K there exists a converging ideal I representing K, i.e., $\varphi(I) = K$. Let **FB** denote the set of formal balls. For any compact set K let

$$\mathscr{F}_K = \left\{ \{F_1, \dots, F_k\} \subseteq \mathbf{FB}\colon\ i = 1, \dots, k,\ \ F_i \cap K \neq \emptyset \text{ and } K \subseteq \bigcup_i F_i \right\}.$$

By Lemma 4.16 the set $\mathscr{F}_K$ is non-empty and contains elements of arbitrarily small maximal radius. For a finite non-empty subset $G = \{H_1, \dots, H_n\}$ of $\mathscr{F}_K$ define

$$M_G = \left\{ \{F_1, \dots, F_n\}\colon\ F_i \in H_i \text{ and } K \cap \bigcap_i F_i \neq \emptyset \right\}.$$

Let

$$I^K = \{B \in \mathscr{P}_f^*(D_\mathrm{c})\colon\ B \sqsubseteq_P M_G,\ G \in \mathscr{P}_f^*(\mathscr{F}_K)\}.$$

Lemma 4.24. *The set I^K is a converging ideal and $\varphi(I^K)=K$.*

Proof. It is clear that I^K is non-empty and closed downwards with respect to $\sqsubseteq_P$. In order to show that I^K is an ideal we show for any $G=\{H_1,\ldots,H_n\}$ and $G'=\{H'_1,\ldots,H'_m\}$ that $M_{G\cup G'}$ is an upper bound of M_G and $M_{G'}$. Assume that $\{F_1,\ldots,F_n\}\in M_G$. By definition of M_G there exists $x\in K$ such that $x\in F_i$ for $i=1,\ldots,n$. For any $H'_i\in G'$, $K\subseteq\bigcup\{F\colon F\in H'_i\}$, hence there exists $F'_i\in H'_i$ such that $x\in F'_i$. Thus

$$\{F_1,\ldots,F_n,F'_1,\ldots,F'_m\}\in M_{G\cup G'}$$

and hence the Hoare condition is satisfied. We omit the straightforward verification of the Smyth condition. We have shown that I^K is an ideal.

By Lemma 4.16 we can for any $\varepsilon>0$ find $H\in\mathscr{F}_K$ such that any $F\in H$ has radius less than ε. Letting $G=\{H\}$ we find witnesses M_G to the convergence of I^K. Moreover, suppose $x\in\varphi(I^K)$. Then, since x is in some F in H, we have that $d(x,K)$ is bounded by 2ε. Since this is true for any $\varepsilon>0$ we have that $x\in K$, i.e., $\varphi(I^K)\subseteq K$.

Let $M_G\in I^K$ and $x\in K$. Then for every $H\in G$ there exists a $F\in H$ such that $x\in F$. Thus there exists an element of M_G witnessing that $x\in\varphi(M_G)$. Hence $K\subseteq\varphi(I^K)$. We have shown $\varphi(I^K)=K$. $\square$

Collecting the results in this section we have the following theorem.

Theorem 4.25. *Let (D,D^{R},ψ) be a standard effective domain representation of a complete effective metric space X and let $E=\mathscr{P}_P(D)$. Let $\varphi: E^{\mathrm{R}}\to\mathscr{P}(X)$ be the function defined in Definition 4.20. Then $(E,E^{\mathrm{R}},\varphi)$ is an effective domain representation of $\mathscr{H}(X)$.*

The representing domain in Theorem 4.25 is an SFP domain and not necessarily a Scott–Ershov domain.

We have so far neglected the topology of the space $\mathscr{H}(X)$ as we did in the previous construction. We will now relate the quotient topology induced by the domain representation on $\mathscr{H}(X)$ to the topology induced by the Hausdorff metric. We will show that they coincide and that the representing function φ is pseudo-open.

Let I and J be converging ideals containing $A=\{\mathbf{F}_{a_1,r_1},\ldots,\mathbf{F}_{a_k,r_k}\}$ and let $s=\max_i 2r_i$. For any $x\in\varphi(I)$ there exists a $\mathbf{F}_{a,r}\in A$ such that $x\in F_{a,r}$. By Corollary 4.23 there exists an $x'\in F_{a,r}\cap\varphi(J)$. Hence $d(x,\varphi(J))\leqslant 2r\leqslant s$. By symmetry $d_H(\varphi(I),\varphi(J))\leqslant s$.

Lemma 4.26. *Let $(E,E^{\mathrm{R}},\varphi)$ be the effective domain representation above of $\mathscr{H}(X)$. Then $\varphi: E^{\mathrm{R}}\to\mathscr{H}(X)$ is continuous.*

Proof. It is sufficient to show that $\varphi^{-1}[B(K,r)]$ is open in E^{R} for an arbitrary basic open set $B(K,r)$. Let $I\in\varphi^{-1}[B(K,r)]$. Since I is converging there exists an $A=\{\mathbf{F}_{a_1,r_1},\ldots,\mathbf{F}_{a_k,r_k}\}\in I$ such that $s=\max_i 2r_i<r-d_{\mathrm{H}}(K,\varphi(I))$. Let J be a converging ideal containing A. Then $d_{\mathrm{H}}(\varphi(I),\varphi(J))\leqslant s<r-d_{\mathrm{H}}(K,\varphi(I))$. Hence $d_{\mathrm{H}}(K,\varphi(J))\leqslant d_{\mathrm{H}}(K,\varphi(I))+d_{\mathrm{H}}(\varphi(I),\varphi(J))<r$. Thus $\uparrow A\cap E^{\mathrm{R}}\subseteq\varphi^{-1}[B(K,r)]$. $\square$

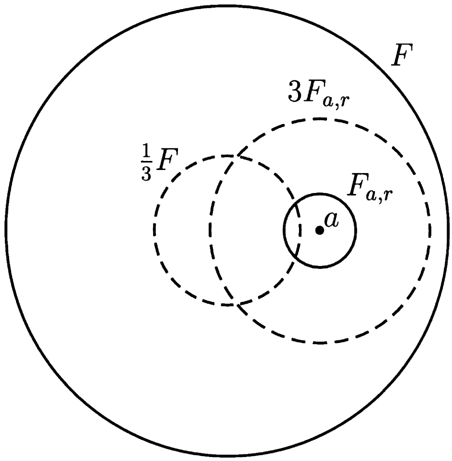

Fig. 1. $3F_{a,r} \preccurlyeq F$.

The ideal I^K is a large ideal in the sense that it contains many approximation of K. We will construct a smaller ideal where the approximations are of a uniform and simple form.

If $F_{a,r}$ is a formal ball then $kF_{a,r}$ denotes the formal ball $F_{a,kr}$. We assume that any set of formal balls covering K is constructed as in Lemma 4.16. In particular, we assume that every formal ball we consider intersects K.

Let K be a compact set. We construct a sequence (A_n) of approximations of K. Let M be a finite set of formal 2^0-balls covering K and let $A_0 = \{3\mathbf{F}\colon F \in M\}$. Suppose A_n satisfies that A_n consists of formal $3 \cdot 2^{-n}$-balls and that K is covered by $\{\frac{1}{3}F\colon \mathbf{F} \in A_n\}$. For every $\mathbf{F} \in A_n$ use Lemma 4.16 to find a finite set M_F of formal 2^{-n-1}-balls covering $K \cap \frac{1}{3}F$. Let

$$A_{n+1} = \bigcup_{\mathbf{F} \in A_n} \{3\mathbf{F}'\colon F' \in M_F\}.$$

Clearly, A_{n+1} consists of formal $3 \cdot 2^{-n-1}$-balls. For any $x \in K$ there exists $\mathbf{F} \in A_n$ such that $x \in \frac{1}{3}F$. Since M_F covers $\frac{1}{3}F \cap K$ there exists an $F' \in M_F$ such that $x \in F'$. Hence $\{\frac{1}{3}F\colon \mathbf{F} \in A_{n+1}\}$ covers K.

We will show that $A_n \sqsubseteq_P A_{n+1}$. Let $F_{a,r} \in M_F$ for some $\mathbf{F} \in A_n$. Then $F_{a,r}$ is a formal 2^{-n-1}-ball where $d(a, K \cap \frac{1}{3}F) < 2^{-n-1}$. Hence $3F_{a,r} \preccurlyeq F$, cf. Fig. 1. Thus the Smyth condition is verified. If $\mathbf{F} \in A_n$ then there exists an $F' \in M_F$ such that $3F' \subseteq F$ since M_F covers $K \cap \frac{1}{3}F \neq \emptyset$. Hence $\mathbf{F}' \in A_{n+1}$ and the Hoare condition is thereby verified.

Let I_K be the converging ideal generated by the chain (A_n). We have that $K \subseteq \varphi(I_K)$ since A_n covers K for $n \in \omega$. If $x \notin K$ then there exists an open sphere around x that avoids K. Hence there exists an $n \in \omega$ such that $x \notin \varphi(A_n) \supseteq \varphi(I_K)$. Thus we have that $K = \varphi(I_K)$. Note that I_K is not uniquely determined by K since there may be many possible ε-nets in a compact set.

Lemma 4.27. *Let $(E, E^{\mathrm{R}}, \varphi)$ be the effective domain representation of $\mathscr{H}(X)$ constructed above. Then φ is a pseudo-open mapping.*

Proof. Let $K \in \mathscr{H}(X)$ and let $U \subseteq E^{\mathrm{R}}$ be an open set such that $\varphi^{-1}[K] \subseteq U$. Then the ideal I_K belongs to U. There exists an $A \in I_K$ such that $\uparrow[A] \subseteq U$. We can choose A to be A_n for some n in the construction of I_K.

Let $K' \in B(K, 2^{-n})$. We will show that $A_n \in I^{K'}$. Then it follows that $K' = \varphi(I^{K'}) \in \varphi[U]$ and hence that $K \in (\varphi[U])^{\circ}$.

Let $y \in K'$. Then there exists an $x \in K$ such that $d(x, y) < 2^{-n}$. Since $x \in K$ there exists an $\mathbf{F} \in A_n$ such that $x \in \frac{1}{3}F$. But F is a formal $3 \cdot 2^{-n}$-ball, hence any point within 2^{-n} from x will belong to F. In particular, $y \in F$. Thus $K' \subseteq \bigcup\{F\colon \mathbf{F} \in A_n\}$.

If $\mathbf{F} \in A_n$ then $\frac{1}{3}F$ intersects K, by assumption on the construction of I_K. Thus there exists an $x \in \frac{1}{3}F \cap K$. There exists a $y \in K'$ such that $d(x, y) < 2^{-n}$. Clearly $y \in F$, i.e., $F \cap K' \neq \emptyset$. This concludes the proof that $A_n \in I^{K'}$. □

Any pseudo-open map is a quotient map, hence the lemma above implies that φ is a quotient map.

Theorem 4.28. *Let $(D, D^{\mathrm{R}}, \psi)$ be a standard pseudo-open effective domain representation of a complete effective metric space X. Then there exists a pseudo-open effective domain representation $(E, E^{\mathrm{R}}, \varphi)$ of the metric space $(\mathscr{H}(X), d_{\mathrm{H}})$, obtained uniformly and effectively from $(D, D^{\mathrm{R}}, \psi)$.*

Proof. By Theorem 4.25 and Lemmas 4.26 and 4.27. □

We introduce two notions of effective compactness and effective total boundedness. The first is induced from the effective domain representation of $\mathscr{H}(X)$ while the other is defined directly from an effective metric space. These two notions are then shown to be effectively equivalent.

Definition 4.29. Let $(E, E^{\mathrm{R}}, \varphi)$ be the effective domain representation of $\mathscr{H}(X)$ constructed above. Then a compact set $K \in \mathscr{H}(X)$ is

(i) *effectively compact* if there exists an effective ideal $I \in E^{\mathrm{R}}$ such that $\varphi(I) = K$;

(ii) *effectively totally bounded* if there exists a computable function $f : \mathbb{Q}_+ \to \mathscr{P}_f(X_k)$ such that $f(\varepsilon)$ is an ε-net in K, where X_k are the computable elements of X.

Theorem 4.30. *A compact set K is effectively compact if, and only if, it is effectively totally bounded. Moreover, an r.e.-index of an effective ideal I representing a compact set K is uniformly computable from an index of the function witnessing that K is effectively totally bounded, and vice versa.*

Proof. Let $\varepsilon > 0$ and let K be effectively compact. Then there exists an effective ideal $I \in E^{\mathrm{R}}$ such that $\varphi(I) = K$. Since I is converging we can effectively find an

$$\begin{tikzcd} D_1^R \arrow[r, "f"] \arrow[d, "\psi_1"'] & D_2^R \arrow[d, "\psi_2"] \\ X_1 \arrow[r, "g"] & X_2 \end{tikzcd} \qquad \begin{tikzcd} E_1^R \arrow[r, dashed, "\bar{f}"] \arrow[d, "\varphi_1"'] & E_2^R \arrow[d, "\varphi_2"] \\ \mathcal{H}(X_1) \arrow[r, "g"] & \mathcal{H}(X_2) \end{tikzcd}$$

Fig. 2. Lifting of a representing function to the powerdomain.

$A=\{\mathbf{F}_{a_1,r_1},\dots,\mathbf{F}_{a_n,r_n}\}\in I$ such that $r_i<\frac{\varepsilon}{3}$. By Lemma 4.21 we can effectively compute $x_i\in K$ such that $d(a_i,x_i)<\frac{\varepsilon}{2}$. Clearly the set $\{x_i\colon 1\leqslant i\leqslant n\}$ is an ε-net in K.

Let K be effectively totally bounded. We claim that it is possible to effectivise the construction of I_K. The only step in the construction of I_K that is not already clearly effective is the use of Lemma 4.16. However, in the presence of a computable function f witnessing that K is effectively totally bounded, it is clearly possible to effectivise this lemma. □

Let $(D_i,D_i^{\mathrm{R}},\psi_i)$ be standard effective domain representations of the effective complete metric spaces X_i, and let $(E_i,E_i^{\mathrm{R}},\varphi_i)$ be the effective domain representations of $\mathscr{H}(X_i)$ constructed as above from $(D_i,D_i^{\mathrm{R}},\psi_i)$ for $i=1,2$.

If $f:D_1\to D_2$ is an effective continuous function then it induces an effective and continuous function $g:X_1\to X_2$. Since the continuous image of a compact set is compact we can view g as a function from $\mathscr{H}(X_1)$ to $\mathscr{H}(X_2)$. We would like to construct an effective function $\bar{f}:E_1\to E_2$ from f such that $\varphi_2(\bar{f}(I))=g(\varphi_1(I))$. Cf. Fig. 2.

Definition 4.31. Let $f:D_1\to D_2$ be a continuous function. Then define a function $\bar{f}:\mathscr{P}_f^*((D_1)_{\mathrm{c}})\to E_2$ by

$$\bar{f}(A)=[\{f(\sigma)\colon\ \sigma\in A\}].$$

Remember the notation $[\cdot]$ introduced after Lemma 2.14. Inside the brackets in the definition of $\bar{f}$ is a finite set of elements from D_2 and hence the bracketed expression denotes an ideal in E_2. Note that if f is an effective function then $\bar{f}$ is an effective function by Lemma 2.15.

Lemma 4.32. *Let $A,B\in\mathscr{P}_f^*((D_1)_{\mathrm{c}})$. Then $A\sqsubseteq_P B\Rightarrow\bar{f}(A)\ \sqsubseteq_P\ \bar{f}(B)$.*

Proof.

$$\begin{aligned} A\sqsubseteq_P B \;&\Leftrightarrow\; (\forall\tau\in B)(\exists\sigma\in A)(\sigma\sqsubseteq\tau) \text{ and}\\ &\qquad (\forall\sigma\in A)(\exists\tau\in B)(\sigma\sqsubseteq\tau)\\ &\Rightarrow\; (\forall\tau\in B)(\exists\sigma\in A)(f(\sigma)\sqsubseteq f(\tau)) \text{ and} \end{aligned}$$

$$\begin{aligned} & (\forall \sigma \in A)(\exists \tau \in B)(f(\sigma) \sqsubseteq f(\tau)) \\ \Leftrightarrow \; & \bar{f}(A) \sqsubseteq_P \bar{f}(B). \end{aligned}$$

The implication is true since f is monotone. □

The lemma above shows that $\bar{f}$ is well defined and monotone and hence we can extend $\bar{f}$ to a continuous function $\bar{f}: E_1 \to E_2$.

We need the following topological fact.

Lemma 4.33. *Let K be a compact subspace of a metric space X and let $(x_n)_{n\in\omega}$ be a sequence in X such that $\lim_n d(x_n, K) = 0$. Then there is an $x \in K$ such that x is a cluster point of the sequence $(x_n)_{n\in\omega}$.*

Proof. We construct a new sequence $(y_m)_{m\in\omega}$ in K. Let $a_1, \ldots, a_k$ be a 2^{-m}-net in K. Consider the open cover $\{B(a_i, 2^{-m+1})\colon 1 \leqslant i \leqslant k\}$. The sequence (x_n) will eventually be included in this cover. Hence the sequence (x_n) will infinitely often be in $B(a_i, 2^{-m+1})$ for some i, let y_m be such an a_i. The sequence (y_m) has a cluster point $x \in K$ since K is sequentially compact. To see that x is also a cluster point of (x_n) we have to show that (x_n) is infinitely often in any open neighbourhood of x. Let $\varepsilon > 0$ and let m be an index such that $d(x, y_m) < \varepsilon/2$ and such that $2^{-m+2} < \varepsilon$. If n is an index such that $d(x_n, y_m) < 2^{-m+1}$ then $d(x_n, x) \leqslant d(x_n, y_m) + d(y_m, x) < 2^{-m+1} + \varepsilon/2 < \varepsilon$. There are infinitely many such indices, so x is a cluster point of the sequence (x_n). □

Lemma 4.34. *If $f: D_1 \to D_2$ is a continuous function such that $f[D_1^{\mathrm{R}}] \subseteq D_2^{\mathrm{R}}$ then $\bar{f}[E_1^{\mathrm{R}}] \subseteq E_2^{\mathrm{R}}$.*

Proof. The proof is by contradiction. We begin by finding a critical point $x \in X_1$. We then use the instance $f(I_x) \in D_2^{\mathrm{R}}$ of the assumption to get a contradiction.

Let $I \in E_1^{\mathrm{R}}$, $K = \varphi_1(I)$ and $J = \bar{f}(I)$, and let the sequence $(A_n)_{n\in\omega}$ witness that I converges. Assume that J fails to converge. Then there exists an $\varepsilon > 0$ such that for all $B \in J$ of the form $B = \{\mathbf{F}_{b_1,s_1}, \ldots, \mathbf{F}_{b_k,s_k}\}$ there exists an $i \in \{1, \ldots, k\}$ such that $s_i > \varepsilon$.

We will select a formal ball from $A_n = \{\mathbf{F}_{a_{n1},r_{n1}}, \ldots, \mathbf{F}_{a_{nk_n},r_{nk_n}}\}$ for every n. If there exists an $\mathbf{F}_{b_i,\varepsilon} \sqsubseteq f(\mathbf{F}_{a_{ni},r_{ni}})$ for every $i \in \{1, \ldots, k_n\}$ then $\{\mathbf{F}_{b_1,\varepsilon}, \ldots, \mathbf{F}_{b_{k_n},\varepsilon}\} \in J$ contradicting that J fails to converge. Hence there exists an $i_n \in \{1, \ldots, k_n\}$ such that $f(\mathbf{F}_{a_{ni_n},r_{ni_n}})$ is not above any singleton set of formal balls of radius less than or equal to ε. Consider the sequence (a_{ni_n}) of centre points of the formal balls selected from A_n, $n \in \omega$. By Corollary 4.23 the sequence (a_{ni_n}) has the property $\lim_n d(a_{ni_n}, K) = 0$. Hence by Lemma 4.33 there exists a cluster point $x \in K$ of the sequence (a_{ni_n}).

Consider the ideal I_x, the smallest ideal converging to x. The function f maps converging ideals to converging ideals. Hence for any $\varepsilon > 0$ there exists $\mathbf{F}_{a,r} \in I_x$ and $\mathbf{F}_{b,\varepsilon} \in f(I_x)$ such that $\mathbf{F}_{b,\varepsilon} \sqsubseteq f(\mathbf{F}_{a,r})$. Since $\mathbf{F}_{a,r} \in I_x$ we have that the distance between a and x is strictly less than r. We can hence choose a sufficiently large index n such that $\mathbf{F}_{a_{ni_n},r_{ni_n}} \sqsupseteq \mathbf{F}_{a,r}$ and such that $x \in F_{a_{ni_n},r_{ni_n}}$. This gives a contradiction since $\mathbf{F}_{b,\varepsilon} \sqsubseteq f(\mathbf{F}_{a_{ni_n},r_{ni_n}})$. □

Theorem 4.35. *Let $(D_i, D_i^{\mathrm{R}}, \psi_i)$ be standard effective domain representations of the effective complete metric spaces X_i, and let $(E_i, E_i^{\mathrm{R}}, \varphi_i)$ be the effective domain representations of $\mathscr{H}(X_i)$, for $i=1,2$. Let $f: D_1 \to D_2$ be a continuous function such that $f[D_1^{\mathrm{R}}] \subseteq D_2^{\mathrm{R}}$ and let $g: X_1 \to X_2$ be the continuous function induced by f. Then $\varphi_2(\bar{f}(I)) = g[\varphi_1(I)]$, i.e., $\bar{f}$ represents $g[\cdot]$.*

Proof. Let $K = \varphi_1(I)$. We will show that $\varphi_2(\bar{f}(I)) = g[K]$.

Let $x \in K$ and let $A \in I$. We have $x \in \varphi_1(A)$ since $x \in K = \varphi_1(I)$. Hence there exists a $\sigma \in A$ such that $x \in \varphi_1(\sigma) = \psi_1[\uparrow\sigma \cap D_1^{\mathrm{R}}]$. Since f represents g it follows that $g(x) \in \psi_2[\uparrow(f(\sigma)) \cap D_2^{\mathrm{R}}] = \varphi_2(f(\sigma)) \subseteq \varphi_2(\bar{f}(A))$. Hence $g[K] \subseteq \varphi_2(\bar{f}(A))$ for every $A \in I$ so $g[K] \subseteq \varphi_2(\bar{f}(I))$.

Let $y \in \varphi_2(\bar{f}(I))$. Since $\bar{f}(I)$ is converging there exists a sequence (B_n) in $\bar{f}(I)$ such that B_n consists of singleton sets of formal balls, where the radii of the formal balls are bounded by 2^{-n}. Since $y \in \varphi_2(\bar{f}(I))$ we have that for each n there exists a formal ball $\mathbf{F}_{b_n,s_n} \in B_n$ such that $y \in F_{b_n,s_n}$. For every n there exists an element $A_n \in I$ such that $B_n \sqsubseteq \bar{f}(A_n)$. There exists an $\mathbf{F}_{a_n,r_n} \in A_n$ such that $\mathbf{F}_{b_n,s_n} \sqsubseteq f(\mathbf{F}_{a_n,r_n})$, which implies $g[F_{a_n,r_n}] \subseteq F_{b_n,s_n}$. By Corollary 4.23 there exists an $x \in K \cap F_{a_n,r_n}$, call it x_n. The sequence $(g(x_n))$ is a sequence in $g[K]$ which converges to y and since $g[K]$ is closed, $y \in g[K]$. Hence $\varphi_2(\bar{f}(I)) \subseteq g[K]$. $\square$

5. Hyperbolic IFS

An *iterated function system* or *IFS* is a finite set of functions operating on a metric space. See [3] for an introduction to this area.

Definition 5.1. A *hyperbolic IFS* consists of a complete metric space and a finite set of contractive functions.

We will immediately consider effectivity.

Definition 5.2. An *effective hyperbolic IFS* consists of an effective complete metric space and a finite set of effective contractive functions.

The *image* of a set A under an IFS is the union of the images under the functions in the system. An *attractor* of an IFS is a subset A of the space which is invariant under the system, i.e., the image of A under the system is A, and such that there exists an open neighbourhood U of A satisfying that the image of U is a subset of U.

In order to prove Theorem 5.4 we need to have representing functions that for any approximation give a fairly good approximation of the image of the approximation.

Lemma 5.3. *Let $(D, D^{\mathrm{R}}, \varphi)$ be a standard effective domain representation of an effective complete metric space X and let $g: X \to X$ be an effective contraction with Lipschitz constant $k<1$. Then there exists an effective function $f: D \to D$*

representing g and such that for each formal ball $F_{a,r}$ and for each rational $s>kr$ there exists a formal ball $F_{b,s}$ such that $\mathbf{F}_{b,s} \in f(\mathbf{F}_{a,r})$.

Proof. Let $h:D\to D$ be an effective function representing g such that $h(I_x)=I_{g(x)}$, for $x\in X$. Such an h can be found as the composition of an effective function representing g and the effective function that for any compact element of D gives the proper approximations of that element. This latter function takes any ideal $I\in D^{\mathrm{R}}$ to I_x if $I\to x$. Let

$$\mathscr{F}_\sigma=\{F_{b,s}\colon (\exists F_{a,r}\in\sigma)(\exists F_{c,t}\in h(I_a))(d(b,c)\leqslant s-t-kr)\}, \tag{3}$$

where $\sigma\in D_{\mathrm{c}}$. Define $f:D_c\to D$ by

$$f(\sigma)=\{\tau\subseteq_f\mathscr{F}_\sigma\colon\ \tau \text{ is permissible}\}.$$

We have that $f(\sigma)$ is downwards closed since $\mathscr{F}_\sigma$ is downwards closed. Let $\tau,\tau'\in f(\sigma)$, then we can find an upper bound of τ and τ' by taking an appropriate subset of $\tau\cup\tau'$. Hence $f(\sigma)$ is an ideal in D. We show that f is monotone. Suppose $\sigma\sqsubseteq\sigma'$. Let $\mathbf{F}_{b,s}\in f(\sigma)$, then there exists $F_{a,r}\in\sigma$ and there exists $\mathbf{F}_{c,t}\in h(I_a)$ such that $d(b,c)\leqslant s-t-kr$. Since $\sigma\sqsubseteq\sigma'$ there exists $F_{a',r'}\in\sigma'$ such that $F_{a',r'}\preccurlyeq F_{a,r}$, i.e., $d(a,a')\leqslant r-r'$. By contractivity $d(g(a),g(a'))\leqslant k(r-r')$. It follows that $\mathbf{F}_{c,t+k(r-r')}\in h(I_{a'})$, since $\mathbf{F}_{c,t}\in h(I_a)$. The formal ball $F_{c,t+k(r-r')}$ witnesses that $F_{b,s}\in\mathscr{F}_{\sigma'}$, since

$$d(b,c)\leqslant s-t-kr=s-(t+k(r-r'))-kr'.$$

We have thereby shown that f is monotone and hence that it can be extended to a continuous function on D. It is clear that f is an effective function representing g.

Since $h(I_a)$ converges there exists $F_{c,t}\in h(I_a)$ for arbitrarily small t. By choosing b in (3) to be c we have that $\mathscr{F}_\sigma$ contains formal balls of any radius greater than kr. $\square$

Theorem 5.4. *An effective hyperbolic IFS on an effective complete metric space X has a unique effective non-empty compact attractor.*

Proof. Let (D,D^{R},ψ) be a standard effective domain representation of X and let $(E,E^{\mathrm{R}},\varphi)$ be the effective domain representation of $\mathscr{H}(X)$ constructed in Section 4.

We are given effective functions $g_1,\ldots,g_n$ on X and rational Lipschitz constants $k_i<1$ for each g_i. Let $f_i:D\to D$ be effective functions representing g_i constructed as in Lemma 5.3.

Construct a function $\bar{f}:E_{\mathrm{c}}\to E$ by

$$\bar{f}(A)=\left\{\bigcup_{i=1}^{n}B_i\colon B_i\in\bar{f}_i(A)\right\}.$$

We will show that $\bar{f}(A)$ is an ideal. Let $C\sqsubseteq_P B$ for some $B\in\bar{f}(A)$. Let $C_i=\{\sigma\in C:\exists\tau\in B_i(\sigma\sqsubseteq\tau)\}$. For each $\sigma\in C$ there exists $\tau\in B$ such that $\sigma\sqsubseteq\tau$. Hence $C=\bigcup_i C_i$. Clearly $C_i\sqsubseteq_P B_i$. Hence $C_i\in\bar{f}_i(A)$. Thus $C=\bigcup_i C_i\in\bar{f}(A)$.

Assume that $B = \bigcup_i B_i$ and $B' = \bigcup_i B'_i$ belong to $\bar{f}(A)$. For each i there exists a $C_i \in \bar{f}_i(A)$ such that $B_i, B'_i \sqsubseteq_P C_i$. Clearly $C = \bigcup_i C_i \in \bar{f}(A)$ is an upper bound of B and B'. Thus $\bar{f}(A)$ is an ideal.

Since $\bar{f}$ is a monotone function it can be extended to a continuous function on E. Let $g : \mathscr{H}(X) \to \mathscr{H}(X)$ be defined by

$$g[K] = \bigcup_{i=1}^{n} g_i[K].$$

For each $B \in \bar{f}(I)$ and $i \in \{1, \dots, n\}$ there exists, by the definition of $\bar{f}$, $B_i \subseteq B$ such that $B_i \in \bar{f}_i(I)$. Clearly, $\varphi(B_i) \subseteq \varphi(B)$, hence $\varphi(\bar{f}_i(I)) \subseteq \varphi(\bar{f}(I))$, and since $\varphi(\bar{f}_i(I)) = g_i[\varphi(I)]$ we have

$$g[\varphi(I)] = \bigcup_i g_i\,[\varphi(I)] = \bigcup_i \varphi(\bar{f}_i(I)) \subseteq \varphi(\bar{f}(I)).$$

Assume $x \notin g_i[\varphi(I)] = \varphi(\bar{f}_i(I))$, for $i \in \{1, \dots, n\}$. For each i there exists $B_i \in \bar{f}_i(I)$ such that $x \notin \varphi(B_i)$. Thus $x \notin \varphi(\bigcup_i B_i)$, and $\bigcup_i B_i \in \bar{f}(I)$, so $x \notin \varphi(\bar{f}(I))$. We have shown that $g[\varphi(I)] = \varphi(\bar{f}(I))$, i.e., $\bar{f}$ represents g.

Remember that a formal ball is centred in a point of some dense subset X_0 of X. Let $a \in X_0$ and let $k = \max_i k_i$. Choose d and r rational such that $d \geqslant \max_i d(g_i(a), a)$ and $r > d/(1-k)$. Note that we can compute k and appropriate choices of d and r uniformly from f_i and k_i. Let $\varepsilon > 0$ be such that $r = (d + \varepsilon)/(1-k)$.

Let s be such that $kr < s \leqslant kr + \varepsilon/2$. By Lemma 5.3 there exists a formal ball $F_{b,s}$ such that $\mathbf{F}_{b,s} \in f_i(\mathbf{F}_{a,r})$. Let $\mathbf{F}_{c,t}$ be the approximation of $g(a)$ in (3) that witnesses that $\mathbf{F}_{b,s} \in f_i(\mathbf{F}_{a,r})$. We have that $\mathbf{F}_{a,r} \sqsubseteq \mathbf{F}_{b,s}$ since

$$\begin{aligned} d(a,b) &\leqslant d(a, g(a)) + d(g(a), c) + d(c, b) \\ &\leqslant d + t + (s - t - kr) \\ &= r - kr - \varepsilon + s - kr \\ &= r - s - \varepsilon + 2(s - kr) \leqslant r - s - \varepsilon + \varepsilon = r - s. \end{aligned}$$

We have shown for every i that there exists a formal ball F_i such that $\mathbf{F}_i \in f_i(\mathbf{F}_{a,r})$ and $\mathbf{F}_{a,r} \sqsubseteq \mathbf{F}_i$. Clearly $\mathbf{F}_i$ is also in $\bar{f}_i(\mathbf{F}_{a,r})$ and hence we have that $\{\mathbf{F}_1, \dots, \mathbf{F}_n\} \in \bar{f}(\mathbf{F}_{a,r})$. Furthermore $\{\mathbf{F}_{a,r}\} \sqsubseteq_P \{\mathbf{F}_1, \dots, \mathbf{F}_n\}$. Thus $\{\mathbf{F}_{a,r}\} \sqsubseteq_P \bar{f}(\{\mathbf{F}_{a,r}\})$. Hence, by the least fixed point theorem for domains, there exists a least fixed point I of $\bar{f}$ above $\{\mathbf{F}_{a,r}\}$. This fixed point can be calculated as

$$\bigsqcup_{k \geqslant 0} \bar{f}^k(\{\mathbf{F}_{a,r}\}).$$

We have to show that I converges. For any formal ball $F_{c,t}$ and for $i = 1, \dots, n$ we have by Lemma 5.3 that there exists a formal ball $F_{b,s}$ such that $\mathbf{F}_{b,s} \in f_i(\mathbf{F}_{c,t})$ and $s < t(1 + k)/2$. Hence there exists a finite set A of formal balls with radii bounded by $t(1 + k)/t$ such that $A \in \bar{f}(\{\mathbf{F}_{c,t}\})$. It follows that for all m there exists $A_m = \{\mathbf{F}_{m1}, \dots, \mathbf{F}_{mp_m}\}$ such that $A_m \in \bar{f}^m(\{\mathbf{F}_{a,r}\})$ and such that the radii of the formal balls is bounded by $((1+k)/2)^m r$. But $((1+k)/2)^m \to 0$ as $m \to \infty$ and hence we have shown that I converges.

We have that $\varphi(I)$ is a fixed point of g since

$$g[\varphi(I)] = \varphi(\bar{f}(I)) = \varphi(I).$$

Moreover, since $\bar{f}$ maps $\{\mathbf{F}_{a,r}\}$ to some element strictly above $\{\mathbf{F}_{a,r}\}$, $F^{\circ}_{a,r}$ is a neighbourhood of $\varphi(I)$ and $g[F^{\circ}_{a,r}] \subseteq F^{\circ}_{a,r}$. Hence $\varphi(I)$ is an attractor of the IFS.

To show uniqueness consider successively larger formal balls centred in a. There exists an invariant set in each of these and they must coincide. $\square$

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