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Domain representations of partial functions, with applications to spatial objects and constructive volume geometry

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Abstract

A partial spatial object is a partial map from space to data. Data types of partial spatial objects are modelled by topological algebras of partial maps and are the foundation for a high level approach to volume graphics called *constructive volume geometry* (CVG), where space and data are subspaces of n dimensional Euclidean space. We investigate the computability of partial spatial object data types, in general and in volume graphics, using the theory of effective domain representations for topological algebras. The basic mathematical problem considered is to classify which partial functions between topological spaces can be represented by total continuous functions between given domain representations of the spaces. We prove theorems about partial functions on regular Hausdorff spaces and their domain representations, and apply the results to partial spatial objects and CVG algebras. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Many mathematical models of physical objects and processes are based on a notion of state that specifies the object or process by assigning some data to each point of physical space. Let X be a topological space representing physical space and let A be a topological space representing data. Then a spatial object is represented by a map

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$\mathbf{o}: X \to A$, where

$$\mathbf{o}(x) = \text{the data characterising the object at point } x \in X.$$

Commonly, X is a subspace of 3-dimensional Euclidean space and A is a subspace of $\mathbb{R}^n$, $n > 0$. The combination of sets of spatial objects and operations on spatial objects create complex data types modelled by topological algebras. We are interested in the computability of these spatial objects and topological algebras.

Now the theory of computable functions on topological spaces, such as $\mathbb{R}^n$, has revealed two conditions on functions that are vital for computable modelling. The first is continuity, since computable functions are continuous (thanks to Ceitin's Theorem). The second is partiality, since continuous total functions from connected spaces into discrete spaces are constant. Thus the typical spatial object is a continuous partial function $\mathbf{o}: X \to_{\mathrm{p}} A$, and the typical operation on partial spatial objects is a partial continuous functional on partial functions.

In this paper we consider topological algebras of partial spatial objects and their application to modelling in volume graphics and visualisation. We investigate computability using the *theory of effective domain representations*.

In this paper we will present a many sorted algebra $O_{\mathrm{p}}(X,A)$ to model the data type of partial spatial objects under the hypothesis that space X and data A are regular Hausdorff spaces. Next, we present a domain-theoretic model of the implementation of the algebra $O_{\mathrm{p}}(X,A)$ of partial spatial objects. This is done by giving domain representations of the spaces X and A and building a domain representation of partial functions $X \to_{\mathrm{p}} A$, and of functionals defined on the domain representable partial functions. Different domain representations of X and A lead to different classes of partial functions being representable.

At the theoretical heart of the paper is the study of representing partial functions. Suppose two topological spaces X and Y have representations by domains D and E, respectively. A discontinuous function $f: X \to Y$ does not have a domain representation. However, sometimes there exists a continuous domain function $\bar{f}: D \to E$ such that $\bar{f}$ represents f at every point where f is continuous. In [8] the problem of representing non-continuous total functions is studied by means of the notion of approximative representations of functions. Some basic results regarding this notions were shown. Here, instead, we consider representations of the *partial continuous* function f' defined from a discontinuous function f by

$$f'(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x, \\ \uparrow & \text{otherwise.} \end{cases}$$

The problem here is to characterise those partial functions from X to Y which can be represented by continuous total functions on the domains D and E.

Next, the algebra $O_{\mathrm{p}}(X,A)$ of spatial objects and its domain representation is applied to problems in the semantic foundations of volume graphics. Indeed, our general algebraic model $O_p(X,A)$ is inspired by a new algebraic approach to volume graphics called *constructive volume geometry*.

Volume graphics techniques originate in visualisations of 3-dimensional data sets. For example, in medical imaging, physical objects are measured by various scanning instruments that produce a data file that approximates a partial spatial object of the form $\mathbf{o} : \mathbb{R}^3 \to A$, where A is the space of attributes of the object. Many operations are needed to create new objects, and to visualise and render the data set.

Now volume graphics is an alternate paradigm for computer graphics in which objects are represented by volumes instead of surface representations: see [21, 11]. The key notion in volume graphics is the voxel, which is commonly and informally defined as follows.

A *voxel* (R,d) is a region R of space together with associated data d which specifies the attributes of a physical object in the region R.

Typically, the voxel is a small cube of $\mathbb{R}^3$ together with approximations of the values of the attributes. The cube and the approximations are finitely representable and, hence, the voxel is a simple finite approximation of the volume object. In practice, volume graphics is based on algorithms that work on these finite approximations, the voxels, for the transformation and rendering of volume objects.

For volume graphics we have the following:

Problem. To develop a semantic framework in order to analyse how
(i) volume objects are the limit of their voxel approximations, and
(ii) operations on volume objects are correctly implemented via computations with voxel approximations.

Constructive volume geometry (CVG) is a new high level approach to volume graphics based on algebras of volume objects of the form $\mathbf{o} : \mathbb{R}^n \to A$ rather than voxels: see [12–14]. In CVG one *starts* by choosing some attributes A and creating algebras $O_p(\mathbb{R}^n, A)$ of spatial objects with operations that can be used to put together images to form complex scenes. The algebra $O_p(X,A)$ is in fact a mathematical generalisation of the wide spectrum of algebras that are the basis of the constructive volume geometry approach to volume graphics. We will focus on the 4-channel algebra which is one of the simplest computer graphics models, where the attributes are measurements of opacity, red, green, and blue.

CVG is a generalisation of *constructive solid geometry* (CSG). In CSG solids are described by characteristic functions $s : \mathbb{R}^3 \to \{\text{true}, \text{false}\}$ and algebras are created to build solid complex objects from simpler components. The technique is well established in CAD applications. The computability of CSG is also a topic of importance and domain representations of the data types of CSG have been considered recently in [17, 18].

When we apply our general constructions to partial volume objects $\mathbf{o} : \mathbb{R}^n \to_p \mathbb{R}^k$ we get a domain representation $\mathscr{V}$ whose compact elements are generated by voxels. This domain $\mathscr{V}$ is uniquely determined by the voxels and we call it the *voxel domain*.

Thus, the domain representation approach provides elegant semantic models for the data types in volume graphics, which are capable of characterising the behaviour of voxel techniques in the limit as the voxels become finer. This is a solution to the theoretical problem in volume graphics stated above.

In Section 2 we discuss some general notions about representing partial functions in concrete models of computation. In Section 3 we recall the method of representing topological spaces by domains and give some theory of representing partial functions. In Section 4 we recall the notion of effective domain representations. In Section 5 we apply domain representations to algebras of partial spatial objects. Then, in Section 6 we study domain representations of CVG algebras and their implications.

This paper is one of a series on using algebraic domains to represent topological algebras; earlier studies include complete local rings [30, 31]; ultra metric algebras [32], locally compact spaces [34], metric spaces [5], and regular Hausdorff spaces [7]. Among application areas considered are process theory [32], synchronous concurrent algorithms [33], stream processing [8], and iterated maps [6]. General accounts are in [28, 29, 34]. A complementary theory based on continuous domains with many other applications has been developed by Edalat [15–19].

Of course, the effectivity of these spatial object algebras and their applications in volume graphics can also be investigated by other models of computation, such as the abstract models of Tucker and Zucker [36, 37] and Brattka [9], and the concrete models of Weihrauch's type 2 enumerability [38, 39], Spreen's computable topological spaces [27], and effective metric space theory [23]. Further approaches to computability on topological spaces include equilogical spaces, partial equivalence relations on T_0 spaces, and modest sets [3, 4, 26]. The equivalence of several concrete models of computability was established in Stoltenberg–Hansen and Tucker [35].

2. Partial functions and their representations

2.1. Partial functions

We will make use of the following notations for various function spaces.

Definition 2.1. Let X and Y be non-empty sets. Let $(X \to_{\mathrm{p}} Y)$ be the set of all partial functions from X to Y and let $(X \to Y)$ be the set of all total functions from X to Y.

We will often be interested in function spaces of continuous functions on topological spaces. Therefore, we need to make precise when a partial function is continuous.

Definition 2.2. A partial function $f : X \to Y$, where X and Y are topological spaces, is *continuous* if the function $f : \mathrm{dom}\, f \to Y$ is continuous, where the topology on $\mathrm{dom}\, f$ is the subspace topology from the topology on X.

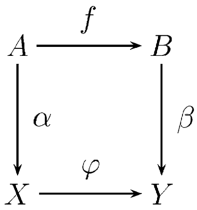

Fig. 1. Representing partial functions.

The above definition amounts to requiring that for any open set $V \subseteq Y$ there exists an open $U \subseteq X$ such that

$$f^{-1}[V] = U \cap \operatorname{dom} f.$$

Definition 2.3. Let X and Y be topological spaces. Let $C_{\mathrm{p}}(X, Y)$ be the set of all continuous partial functions from X to Y and let $C(X, Y)$ be the set of all continuous total functions from X to Y.

2.2. Representations of partial functions

We start by giving the general picture of representations of partial functions on sets. Let A and B be representations of the sets X and Y, respectively, with the representing (total, onto) functions $\alpha : A \to X$ and $\beta : B \to Y$. An element $a \in A$ such that $\alpha(a) = x$ is a *representation* of x.

Definition 2.4. A partial function $\varphi : X \to_{\mathrm{p}} Y$ is *represented* by the partial function $f : A \to_{\mathrm{p}} B$ if
(i) $\alpha^{-1}[\operatorname{dom} \varphi] \subseteq \operatorname{dom} f$, and
(ii) for all $x \in \operatorname{dom} \varphi$ and a a representation of x, $\beta f(a) = \varphi(x)$.

The above definition says that if the function φ is defined for some element x then each representation of x should satisfy the commutative diagram of Fig. 1. If φ is total, then the above definition coincides with the usual definition for representations of total functions.

Note that the representing function may be defined for an $a \in A$ even though $\alpha(a)$ does not belong to the domain of the function φ. This may be interpreted as an erroneous answer in some cases. The following definition gives a stronger form of representation for partial functions.

Definition 2.5. A representation $f : A \to_{\mathrm{p}} B$ of $\varphi : X \to_{\mathrm{p}} Y$ is *true* if

$$\alpha^{-1}[\operatorname{dom} \varphi] = \operatorname{dom} f.$$

The importance of true representations is that the domain of the represented function can be deduced from the representation. If f is a true representation of the partial function φ then φ is uniquely determined, and φ may therefore be said to be *induced* by f.

A function $f : A \to_p B$ is not necessarily a true representation of any partial function $\varphi : X \to_p Y$ since $\operatorname{dom} f$ may contain a proper non-empty subset of $\alpha^{-1}[x]$ for some $x \in X$.

Recall the Kleene equality $\simeq$ defined by $f(x) \simeq g(y)$ if, and only if, both expressions are undefined, or both expressions are defined and equal.

Lemma 2.6. *A function $f : A \to_p B$ is a true representation of a unique partial function if, and only if, f satisfies for every $a, b \in A$*

$$\alpha(a) = \alpha(b) \Rightarrow \beta f(a) \simeq \beta f(b).$$

Proof. Given that f is a true representation of φ, then by definition of being a representation the condition must hold for all $a, b \in \alpha^{-1}[\operatorname{dom} f]$. Since f is a true representation, $f(a)$ is undefined for all $a \notin \alpha^{-1}[\operatorname{dom} f]$.

For the converse, define $\varphi : X \to Y$ by $\varphi(x) \simeq \beta f(a)$, where a is some representation of x. Clearly, the condition implies that φ is well defined and that f is a true representation of φ. □

If X is a topological space, then $\alpha : A \to X$ induces a topology on A, the smallest topology making α continuous. The open sets in A are $\alpha^{-1}[U]$, where $U \subseteq X$ is open, so α will be a quotient map.

Lemma 2.7. *Let X and Y be topological spaces, and let $f : A \to_p B$ represent $\varphi : X \to_p Y$. Then φ is continuous if f is continuous with respect to the topologies induced by α and β.*

Proof. Let V be an open subset of Y. The pre-image $\beta^{-1}[V]$ is open in B. Since f is a partial continuous function we have $f^{-1}[\beta^{-1}[V]]$ is open in $\operatorname{dom} f$. Hence, there exists an open $U \subseteq X$ such that

$$\alpha^{-1}[U] \cap \operatorname{dom} f = f^{-1}[\beta^{-1}[V]].$$

We claim that $\varphi^{-1}[V] = U \cap \operatorname{dom} \varphi$, and hence that φ is a partial continuous function.

Clearly, $\alpha^{-1}[U \cap \operatorname{dom} \varphi] \subseteq \alpha^{-1}[U] \cap \operatorname{dom} f = f^{-1}[\beta^{-1}[V]]$ since f is a representation of φ. Thus, $U \cap \operatorname{dom} \varphi \subseteq \varphi^{-1}[V]$.

From $\alpha^{-1}[\varphi^{-1}[V]] \subseteq f^{-1}[\beta^{-1}[V]] = \alpha^{-1}[U] \cap \operatorname{dom} f$ we get $\varphi^{-1}[V] \subseteq U$, and hence $\varphi^{-1}[V] \subseteq U \cap \operatorname{dom} \varphi$. □

Lemma 2.8. *Let X and Y be topological spaces, and let $f : A \to_p B$ be a true representation of $\varphi : X \to_p Y$. Then f is continuous with respect to the topologies induced by α and β whenever φ is continuous.*

Proof. Let V be an open subset of B. There exists V' open in Y such that $\beta^{-1}[V'] = V$. The pre-image $\varphi^{-1}[V']$ is open in $\operatorname{dom}\varphi$, i.e., there exists an open $U' \subseteq X$ such that $\varphi^{-1}[V'] = U' \cap \operatorname{dom}\varphi$. The set $U = \alpha^{-1}[U']$ is open in A. Since f is a true representation of φ we have $\operatorname{dom} f = \alpha^{-1}[\operatorname{dom}\varphi]$. Using that f represents φ we have $f^{-1}[V] = U \cap \operatorname{dom} f$. Thus, f is a partial continuous function. □

3. Domain representation of partial functions

A domain representation of a topological space is, vaguely, an embedding of that space into a domain. Domain representations exist in several versions and have different properties. The first part of this section discusses the notion of domain representation and the second part concerns the representation of partial functions. For the basic theory of domains we refer to [29]. We will use the notation of [29, 34, 7].

3.1. Domain representations

In [34] a *domain representation* of a topological space X is defined to be a triplet $(D, D^{\mathrm{R}}, \rho)$, where D is a domain, D^{R} is a subset of D and $\rho : D^{\mathrm{R}} \to X$ is a quotient map. They consider (as will we) domain representations where D is an algebraic domain. Representations using continuous domains have also been studied by various authors.

With the weak notion introduced above it can be shown that any T_0 space has a domain representation [7]. Several strengthenings of the notion of domain representations are possible. An *upwards-closed* domain representation is a representation $(D, D^{\mathrm{R}}, \rho)$ where D^{R} is an upper set of D and for every x and y in D

$$\text{if } x \sqsubseteq y \quad \text{and} \quad x \in D^{\mathrm{R}} \text{ then } y \in D^{\mathrm{R}} \quad \text{and} \quad \rho(x) = \rho(y).$$

Upwards-closed representations are intuitive as they make sense when the domains are interpreted as information theoretic structures. The elements of the domain D may be seen as information about a point in the space X. The condition of upwards-closed can then be restated as: if enough information is known to uniquely determine a point in X then adding any consistent information to this should still determine the same point.

A *dense* domain representation $(D, D^{\mathrm{R}}, \rho)$ is a representations where D^{R} is dense in D. A *retract* domain representation $(D, D^{\mathrm{R}}, \rho, \eta)$ is a domain representation $(D, D^{\mathrm{R}}, \rho)$ and a continuous map $\eta : X \to D^{\mathrm{R}}$ such that $\rho \circ \eta = \mathrm{id}_X$. Both the density and retract property are important in lifting continuous functions between topological spaces to their domain representations.

Our standard kind of representations will be dense upwards-closed retract representations.

The following theorem from [7] captures nicely the class of topological spaces that have domain representations of this kind.

Theorem 3.1. *A topological space X has an upwards closed retract domain representation if, and only if, X is a regular Hausdorff space.*

The construction of domain representations used in the proof of the above result will be referred to as standard domain representations. The representations are built from neighbourhood systems satisfying certain conditions.

The following result is also from [7].

Theorem 3.2. *Let D be a dense domain representation of X and E be a retract representation of Y. Then any total continuous function $\varphi : X \to Y$ is represented by a continuous domain function $f : D \to E$.*

Standard representations are in fact dense. Thus, the standard construction of domain representations can be seen as a functor from the category of regular Hausdorff spaces with continuous maps to the category of (retract) domain representations with continuous maps.

The following example presents the interval domain representation of the reals. The interval domain is a standard domain representation $\mathscr{R} = (\mathscr{R}, \mathscr{R}^{\mathrm{R}}, \rho, \eta)$. The example does not suppose any prior knowledge of domains or domain representations. Readers familiar with the interval domain can safely skip this example.

Example 3.3 (*Domain representation of the real line*). Let P be the set of all nonempty closed rational intervals, i.e.,

$$P = \{[a, b]:\ a, b \in \mathbb{Q} \cup \{-\infty, \infty\} \text{ and } a \leqslant b\}.$$

The elements of P are finite approximations of points on the real line. Let P be ordered by $\sqsubseteq$, where $\sqsubseteq$ is defined as reverse inclusion,

$$[a, b] \sqsubseteq [c, d] \quad \Leftrightarrow \quad [a, b] \supseteq [c, d].$$

Two elements in P are *consistent* if they are bounded with respect to the ordering $\sqsubseteq$. That is, two intervals are consistent if they overlap. The consistency predicate is denoted Cons.

The supremum (least upper bound), $\sqcup$, of two consistent intervals is their intersection,

$$[a, b] \sqcup [c, d] = [a, b] \cap [c, d].$$

The structure

$$P = (P; \sqsubseteq, \mathrm{Cons}; \sqcup; \perp)$$

is a conditional upper semi-lattice (cusl), i.e., every finite consistent set has a supremum (the intersection of the intervals).

The concept of ideal is a generalisation of the concept of a sequence. Since the elements of P are approximations, we may say that ideals are generalisations of sequences of approximations. A subset I of P is an *ideal* if

(i) $[a,b]\in I$ and $[c,d]\sqsubseteq[a,b]$ implies $[c,d]\in I$, and
(ii) $[a,b],[c,d]\in I$ implies that there exists $[e,f]\in I$ such that $[a,b]\sqsubseteq[e,f]$ and $[c,d]\sqsubseteq[e,f]$.

One may complete the cusl P by taking the ideal completion $\mathscr{R}=\mathrm{Idl}(P)$, i.e., the set of all ideals over P ordered by set theoretic inclusion. The ideal completion $\mathscr{R}$ is a *domain*. The cusl is isomorphic to a subset of $\mathscr{R}$, this set is known as the set of *compact elements* of the domain, hence the compact elements of $\mathscr{R}$ will be the rational intervals. Thus, a domain contains proper approximations, the compact elements, as well as complete (or total) elements. A *total* element in our setting is an element representing a point in $\mathbb{R}$, i.e., uniquely determining the point.

An ideal $I\in\mathscr{R}$ represents an element $x\in\mathbb{R}$ if $\bigcap I=\{x\}$. The subset of ideals representing some real point are denoted by $\mathscr{R}^{\mathrm{R}}$. The map taking an ideal in $\mathscr{R}^{\mathrm{R}}$ to the element it represents is denoted by ρ. The map $\eta:\mathbb{R}\to\mathscr{R}^{\mathrm{R}}$ is defined by

$$\eta(x)=I_x=\{[a,b]\in P\colon\ a<x<b\}.$$

We note that $\eta(x)$ is the least ideal representing x. Both ρ and η are continuous maps and $\rho\eta=\mathrm{id}_{\mathbb{R}}$ so $\mathbb{R}$ is a retract of $\mathscr{R}^{\mathrm{R}}$.

Retract domain representations are closed under products. Thus, the product $\mathscr{R}^n$ is a domain representation of $\mathbb{R}^n$.

The following lemma establishes a property of $\mathscr{R}$ that we will use later.

Lemma 3.4. *The set $\mathscr{R}^{\mathrm{R}}$ is a G_δ-subset of $\mathscr{R}$.*

Proof. Let $U_n=\uparrow\{[a,a+2^{-n}]\colon\ a\in\mathbb{Q}\}$. Clearly, U_n is open in $\mathscr{R}$ and $\mathscr{R}^{\mathrm{R}}=\bigcap_n U_n$. $\square$

3.2. Domain representations and spaces of partial functions

This subsection develops a theory for representations of partial functions and operations on partial functions. In particular, domain representations of such objects are studied.

3.2.1. Domain representations of partial functions

In this section we consider the problem of representing partial functions between spaces that have domain representations. For the rest of the section, let $(D,D^{\mathrm{R}},\rho_X,\eta_X)$ and $(E,E^{\mathrm{R}},\rho_Y,\eta_Y)$ be retract domain representations of the topological spaces X and Y, respectively.

The functions considered on the domains are always total, and, in fact, continuous. However, the functions that are represented may be partial.

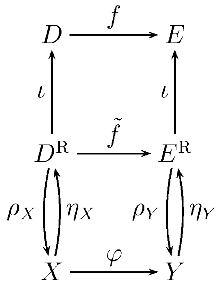

Fig. 2. Domain representation of a partial function φ.

In order to capture the partiality of the functions the domains will have a notion of totality which will correspond to defined values. In our case an element of the domain is *total* if, and only if, it represents some element, i.e., if it belongs to the set of representing elements.

Notions of totality have been studied by Berger [2] and Normann [24] among others.

Since any domain function $f : D \to E$ is assumed to be total, the restriction $f|_{D^{\mathrm{R}}}$ of f to D^{R} is still a total function from D^{R} to E. However, the function f may also be seen as a partial function from D^{R} to E^{R}. Formally, we define the *partialisation* $\tilde{f} : D^{\mathrm{R}} \to_{\mathrm{p}} E^{\mathrm{R}}$ of f by

$$\tilde{f}(d) = \begin{cases} f(d) & \text{if } f(d) \in E^{\mathrm{R}}, \\ \uparrow & \text{otherwise,} \end{cases}$$

where $d \in D^{\mathrm{R}}$. In other words, $\tilde{f}$ is the subgraph of f obtained by restricting to both D^{R} and E^{R}, cf. Fig. 2. Ideally, the notation should convey the subsets, to which the function is restricted. However, this results in a very cumbersome notation, so the reader is trusted to deduce these subsets from the context.

Recall the general notions of representations of partial functions from Definitions 2.4 and 2.5.

Definition 3.5. Let $\varphi : X \to Y$ be a partial function, $f : D \to E$ be a continuous function, and let $\tilde{f} : D^{\mathrm{R}} \to_{\mathrm{p}} E^{\mathrm{R}}$ be the partialisation of f. Then f is a (*true*) *domain representation* of φ, $\mathrm{Repr}(f, \varphi)$, if $\tilde{f}$ is a (true) representation of φ.

By Lemma 2.6, a domain function $f : D \to E$ *induces* a partial function φ if, and only if, for every $d, d' \in D^{\mathrm{R}}$ such that $\rho_X(d) = \rho_X(d')$ it is either the case that
(i) $f(d), f(d') \in E^{\mathrm{R}}$ and $\rho_Y f(d) = \rho_Y f(d')$, or
(ii) $f(d) \notin E^{\mathrm{R}}$ and $f(d') \notin E^{\mathrm{R}}$.
If φ is induced by f then f is a true representation of φ.

The following result gives a sufficient condition for liftings of partial functions to exist.

Theorem 3.6. *Let (D, D^R, ρ_X, η_X) be a dense representation of X, (E, E^R, ρ_Y, η_Y) be a representation of Y, where $E^R \neq E$, and let $\varphi : X \to_p Y$ be a partial continuous function defined on an open subset of X. Then there exists a true representation $f : D \to E$ of φ.*

If $E^R = E$ then the representation of Y may be replaced by the domain $E_\perp$, where $E_\perp^R = E$. Hence, the requirement is no real limitation.

Proof. The construction of f is done in three steps. First, define f' by

$$f' = \eta_Y \varphi \rho_X.$$

By hypothesis, this is a partial continuous function from D^R to E^R, which is a true representation of φ since

$$\operatorname{dom} f' = \rho_X^{-1}[\operatorname{dom} \varphi]$$

and

$$\rho_Y f'(d) = \rho_Y \eta_Y \varphi \rho_X(d) = \varphi \rho_X(d),$$

whenever $d \in \operatorname{dom} f'$. The function f' may be considered as a partial continuous function from D^R to E as well, since E^R has the subspace topology induced from E.

Secondly, extend f' to a total function $f'' : D^R \to E$ by

$$f''(d) = \begin{cases} f'(d) & \text{if } d \in \operatorname{dom} f', \\ \perp & \text{otherwise.} \end{cases}$$

Clearly, the restriction of f'' to a partial function from D^R to E^R is still a true representation of φ, since $\perp \notin E^R$. The function f'' is continuous since $f''^{-1}[E \setminus \{\perp\}] = \operatorname{dom} f' = \rho_X^{-1}[\operatorname{dom} \varphi]$ is open and the only open set of E containing $\perp$ is E itself.

The continuous function $f'' : D^R \to E$ maps a dense subset of D into an injective space E. Hence, the function f'' has an extension to all of D, see, for example, Escardo [20]. The extension f is given by

$$f(d) = \bigsqcup \{\sqcap f''[\uparrow a \cap D^R] : \ a \in \operatorname{approx}(d)\}. \qquad \square$$

The result below shows that an open domain of definition for a partial function sometimes is a necessary condition.

Proposition 3.7. *Let (D, D^R, ρ_X, η_X) and (E, E^R, ρ_Y, η_Y) be domain representations of X and Y respectively. If E^R is open then any partial function $\varphi : X \to Y$ with a true domain representation has an open domain.*

Proof. The set $\rho_X^{-1}[\operatorname{dom} \varphi] = f^{-1}[E^R] \cap D^R$ is open since f is continuous. Thus, since ρ_X is a quotient, $\operatorname{dom} \varphi$ is open. $\square$

For example, the above result applies to the case of partial functions into the booleans, when the booleans is given its usual domain representation $\mathbb{B}_\perp$. Observe that the choice of representation is important. However, the result does not apply to our standard representation $\mathscr{R}$ of the reals. A weaker necessary condition for $\mathscr{R}$ is given in the following result.

Lemma 3.8. *Let $(D, D^{\mathrm{R}}, \rho, \eta)$ be a domain representation of the topological space X. Assume that $\varphi : X \to \mathbb{R}$ is induced by a continuous domain function $f : D \to \mathscr{R}$. Then φ is continuous with domain of definition a G_δ subset of X.*

Proof. By Lemma 3.4 the set $\mathscr{R}^{\mathrm{R}}$ is G_δ. Hence, $\operatorname{dom}\varphi = \eta^{-1}[f^{-1}[\mathscr{R}^{\mathrm{R}}]]$ is G_δ. □

The following example shows that true representations for continuous partial functions sometimes exist even for functions defined on a non-open set.

Example 3.9. Let $\mathbb{R}$ have the usual domain representation $\mathscr{R}$, see Example 3.3. Consider the function $\varphi : \mathbb{R} \to_{\mathrm{p}} \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1, \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that $\operatorname{dom}\varphi = [0,1]$ is a G_δ set. Let $f : \mathscr{R}_{\mathrm{c}} \to \mathscr{R}$ be defined by

$$f([a,b]) = [\min(a,1), \max(0,b)].$$

Then f is a monotone function and therefore extends uniquely to a continuous function $f : \mathscr{R} \to \mathscr{R}$, which is a true representation of φ.

For any ideal representing an element in the closed unit interval, the function f will return a representation of that element. For elements outside that interval, the result is an element not in $\mathscr{R}^{\mathrm{R}}$.

Furthermore, f is an effective function.

3.2.2. The space of representable partial functions

We will here consider the space of represented partial functions. Note that we do not give a domain representation of this space in the sense used in Section 3.1.

Definition 3.10. Let $(D, D^{\mathrm{R}}, \rho_X, \eta_X)$ and $(E, E^{\mathrm{R}}, \rho_Y, \eta_Y)$ be domain representations of the topological spaces X and Y, respectively. The set $\mathrm{Rep}_{\mathrm{p}}^{D,E}(X,Y)$ consist of all representable partial continuous functions from X to Y, i.e.,

$$\mathrm{Rep}_{\mathrm{p}}^{D,E}(X,Y) = \{\varphi \in C_{\mathrm{p}}(X,Y) \colon\ (\exists f \in [D \to E])\, \mathrm{Repr}(f,\varphi)\}.$$

We will usually drop the domains D and E from the notation when clear from the context.

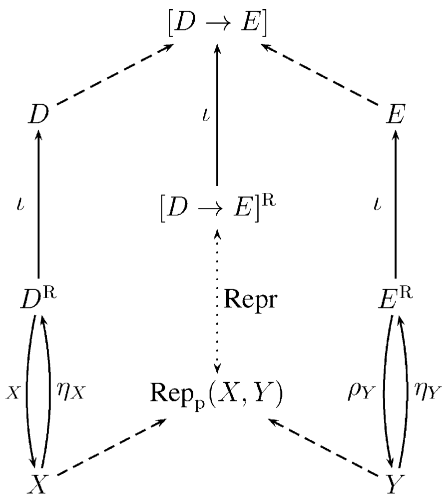

Fig. 3. Domain representations of partial functions.

The domain representation of partial functions is depicted in Fig. 3. We choose the representing elements of $[D \to E]$ to be the whole set, i.e., $[D \to E]^{\mathrm{R}} = [D \to E]$. Theorem 3.6 says that $\mathrm{Rep}_{\mathrm{p}}(X, Y)$ contains all continuous partial functions defined on an open subset of X, in particular, all continuous total functions belong to $\mathrm{Rep}_{\mathrm{p}}(X, Y)$ when the domain representation of X is dense.

$\mathrm{Rep}_{\mathrm{p}}(X, Y)$ is not domain represented in the strong sense of Section 3.1. Neither the retraction, mapping domain functions to partial functions, nor the embedding, mapping partial functions to domain functions, have been defined. However, true representations may be mapped to the partial functions they induce, and continuous partial functions defined on an open set may be mapped to a representation. Neither of these restrictions seems to be viable in our setting. The representation of partial functions is to be viewed as a relation between representatives and objects rather than as a mapping from representatives to objects.

We now define what it means for an operation on $\mathrm{Rep}_{\mathrm{p}}$-spaces to be represented.

Definition 3.11. Let $D, E, E_1, \ldots, E_n$ be retract domain representations of $X, Y, Y_1, \ldots, Y_n$, respectively. An operation

$$\Phi : \mathrm{Rep}_{\mathrm{p}}(X, Y_1) \times \cdots \times \mathrm{Rep}_{\mathrm{p}}(X, Y_n) \to \mathrm{Rep}_{\mathrm{p}}(X, Y)$$

is *represented* by a domain function

$$F : [D \to E_1] \times \cdots \times [D \to E_n] \to [D \to E]$$

if

$$\mathrm{Repr}(f_1, \varphi_1), \ldots, \mathrm{Repr}(f_n, \varphi_n) \Rightarrow \mathrm{Repr}(F(f_1, \ldots, f_n), \Phi(\varphi_1, \ldots, \varphi_n)).$$

4. Effective domains

We will use the natural computability theory for domains to induce a computability theory on the represented spaces. See [29] for a thorough account of the effective theory of domains.

Definition 4.1. (i) A domain D is *effective* if the algebra $D_c = (D_c, \sqsubseteq, \mathrm{Cons}, \sqcup, \bot)$ is a computable algebra.

(ii) A domain representation (D, D^R, ρ, η) is *effective* if D is effective.

(iii) Let $D = (D, D^R, \rho, \eta)$ be an effective domain representation of X. An element $x \in X$ is *computable* if there exist a computable element $d \in D$ representing x.

Example 4.2 (*Effective representation of the real line*). Let $\mathscr{R} = (\mathscr{R}, \mathscr{R}^R, \rho, \eta)$ be the interval domain representation of the reals $\mathbb{R}$ from Example 3.3. Then $\mathscr{R}$ is an effective representation of the reals. Deciding the order on the intervals, $\mathscr{R}_c$, reduces to deciding whether one rational interval is included in another rational interval, which clearly is computable. It is also easy to decide whether two rational intervals are consistent (overlap). If two rational intervals are consistent, then it is easy to compute the supremum (the intersection).

Definition 4.3. (i) A function $f : D \to E$ is *effective* if the set $\{(a, b) \subseteq D_c \times E_c : \ b \sqsubseteq f(a)\}$ is semidecidable.

(ii) A function $\varphi : X \to_p Y$ is *effectively represented* by a domain function $f : D \to E$ if f is effective and represents φ.

Several results on the effectivity theory induced on the reals (and other locally compact spaces) can be found in [34]. For example, the set of reals that are computable is exactly the set of recursive reals, and the notions of effectively represented function, and of computable function, in the Grzegorczyk sense, coincides. A generalisation to metric spaces can be found in [5].

5. Algebras of spatial objects

A *partial spatial object* is an object "residing" in some space X. The modelling process starts with choosing a set of properties, called *attributes*, such that the objects are determined by them, i.e., objects are described extensionally. The attributes may describe, for example, whether the object occupies a point in space or the colour of the object at that point or any other property such as temperature, pressure and so on. Each attribute is given by a partial function that assigns a value from a data set to a point.

The space X in which the partial spatial objects reside may be any topological space. However, we will usually assume that it is some Euclidean space $\mathbb{R}^n$. The data spaces from which the attributes take their values are arbitrary spaces.

Definition 5.1. Let A be a topological space of data and let X be a topological space. A *partial A-scalar field on X* is a partial function $\sigma : X \to_{\mathrm{p}} A$. If σ is total, then σ is a *total A-scalar field on X*.

The prefix denoting the data set is generally omitted if it is clear from the context. A set of partial A-scalar fields on X, i.e., a subset of $(X \to_{\mathrm{p}} A)$, will be denoted by $S_{\mathrm{p}}(X, A)$, where the 'p' is used to point out that scalar fields may be partial mappings. A set of *total* scalar fields is denoted by $S_{\mathrm{t}}(X, A)$.

Definition 5.2. Let $A = (A_1, \ldots, A_k)$ for some data sets A_i. A *partial A-spatial object in X* is a tuple of partial scalar fields

$$\mathbf{o} = (\sigma_1, \ldots, \sigma_k),$$

where $\sigma_i \in (X \to_{\mathrm{p}} A_i)$, for $1 \leqslant i \leqslant k$. A spatial object is *total* if each of the scalar fields is total.

The partial scalar fields constituting a partial spatial object define the *attributes* of the partial spatial object. A set of partial A-spatial objects in X is denoted by

$$O_{\mathrm{p}}(X, A) = \prod_{1 \leqslant i \leqslant k} S_{\mathrm{p}}(X, A_i),$$

where $A = (A_1, \ldots, A_k)$ and A_i is the data set of the ith attribute.

We have preferred to view a partial spatial object as a tuple of partial scalar fields rather than as a partial function from the space into the cartesian product of the data sets. The former allows for some of the attributes to be defined at a point even though others are undefined at that point without having to introduce partially defined tuples.

Having introduced our main type of objects for this paper we can now state that we are interested in studying algebras of the following kind:

algebra	partial spatial objects
carriers	$O_{\mathrm{p}}(X, A)$
operations	$F : O_{\mathrm{p}}(X, A)^m \to O_{\mathrm{p}}(X, A)$
	$\vdots$

The operation F takes m partial spatial objects and returns a new partial spatial object, hence it may be viewed as a single sorted algebra. However, most operations on this algebra will be defined from operations on the underlying partial scalar fields. Thus, an algebra of partial spatial objects will be described as a many sorted algebra later on.

5.1. Structure of algebras of spatial objects

In this section we will consider algebras of partial spatial objects. The algebras will normally contain only operations derived from operations on the partial scalar fields

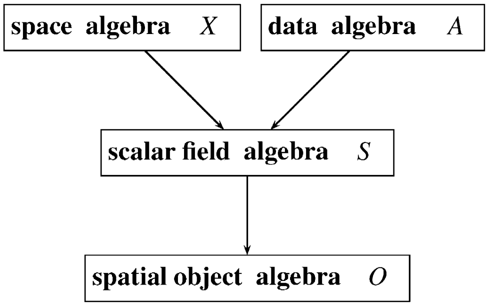

Fig. 4. The algebras used to construct the algebra of partial spatial objects.

by means of projections and tupling. Hence, the algebras are many sorted, constructed from algebras depicted in Fig. 4. We will start by describing the component algebras and move on to the algebras over partial spatial objects later on.

5.1.1. Space and data algebras

The space is a topological space and it is not necessarily equipped with any operations. However, if the space happens to be a metric space, then the metric can be included as an operation from the space to the reals. Moreover, the algebra may also contain operations t transforming the space, for example, affine maps on euclidean spaces. These operations may have parameters from some space P. Such algebras have the form

algebra	$\mathscr{X}$
import	$\mathbb{R}$
carriers	X, P
operations	$d : X^2 \to \mathbb{R}$
	$t : P \to X \to X$
	$\vdots$

The data sets together with their operations form a many sorted algebra.

algebra	$\mathscr{A}$
carriers	$A_1, \ldots, A_n$
operations	$\varphi : A_{s_1} \times \cdots \times A_{s_k} \to A_s$
	$\vdots$

5.1.2. Scalar field algebras

General operations on scalar fields are very unstructured. We will therefore identify two important classes of operations on scalar fields derived from operations on the space and data algebra, respectively. A template for an algebra over scalar fields is the following algebra.

$$\begin{array}{rl}
\textbf{algebra} & \mathscr{S} \\
\textbf{import} & \mathscr{X}, \mathscr{A} \\
\textbf{carriers} & S_{\mathrm{p}}(X, A_1), \ldots, S_{\mathrm{p}}(X, A_n) \\
\textbf{operations} & \mathrm{eval} : S_{\mathrm{p}}(X, A_s) \times X \to A_s \\
 & \bar{\varphi} : S_{\mathrm{p}}(X, A_{s_1}) \times \cdots \times S_{\mathrm{p}}(X, A_{s_k}) \to S_{\mathrm{p}}(X, A_s) \\
 & \bar{t} : P \to S_{\mathrm{p}}(X, A_s) \to S_{\mathrm{p}}(X, A_s) \\
 & \vdots
\end{array}$$

The operation $\bar{\varphi}$ denotes the pointwise extension of a data operation φ. The operation $\bar{t}$ denotes the spatial transformation derived from an operation t in the space algebra.

Let $\mathscr{X}$ be a space algebra and $\mathscr{A}$ a data algebra. Pointwise extensions of operations in the data algebra form one important class of operations on scalar fields.

Definition 5.3. A partial operation $\varphi : A_{s_1} \times \cdots \times A_{s_k} \to_{\mathrm{p}} A_s$ is *extended pointwise* to an operation $\bar{\varphi} : S_{\mathrm{p}}(X, A_{s_1}) \times \cdots \times S_{\mathrm{p}}(X, A_{s_k}) \to_{\mathrm{p}} S_{\mathrm{p}}(X, A_s)$ by

$$\bar{\varphi}(\sigma_1, \ldots, \sigma_k)(x) \simeq \varphi(\sigma_1(x), \ldots, \sigma_k(x)).$$

Pointwise extensions of discontinuous data operations will in general result in discontinuous scalar fields. We will therefore make our data operations continuous by making them partial.

Example 5.4. Consider the data types of reals $\mathbb{R}$ and booleans $\mathbb{B}$. The operation $\leqslant\ : \mathbb{R}^2 \to \mathbb{B}$ is discontinuous. Hence, the pointwise extension $\bar{\leqslant} : S_{\mathrm{p}}(X, \mathbb{R}) \times S_{\mathrm{p}}(X, \mathbb{R}) \to S_{\mathrm{p}}(X, \mathbb{B})$ defined by

$$(\sigma_1 \bar{\leqslant} \sigma_2)(x) \simeq \sigma_1(x) \leqslant \sigma_2(x),$$

will in general give discontinuous scalar fields as results. Let the partial version $\leqslant_{\mathrm{p}}$ of the ordering be defined by

$$\leqslant_{\mathrm{p}} (x, y) = \begin{cases} \text{true} & \text{if } x < y, \\ \text{false} & \text{if } x > y, \\ \uparrow & \text{if } x = y. \end{cases}$$

The pointwise extension $\bar{\leqslant}_{\mathrm{p}}$ will always result in continuous (although partial) scalar fields when the arguments are continuous scalar fields.

The other important class of operations on scalar fields are the spatial transformations, which, e.g., includes translations, scalings, and rotations.

Definition 5.5. Let $\sigma \in S_{\mathrm{p}}(X,A)$ be a partial scalar field and let $t:X \to X$ be a total function. Then the *spatial transformation* of σ under t is the partial scalar field

$$\bar{t}(p)(\sigma)(x) \simeq \sigma(t(p)(x)),$$

where p is the parameter and x is a point in X.

5.1.3. Operations on partial spatial objects

We will assume that any algebra over partial spatial objects contains operations for projecting out partial scalar fields from partial spatial objects (π_i) and for constructing partial spatial objects from partial scalar fields (π). Other primitive operations may be added to this algebra. However, we will not add any further primitive operations in our examples. The operations will instead be generated as terms over the algebra. The operations possible are hence the operations obtained by combining projection and tupling with operations on the partial scalar fields.

algebra	$\mathscr{O}$
import	$\mathscr{S}$
carriers	$O_{\mathrm{p}}(X,A)$
operations	$\pi_i : O_{\mathrm{p}}(X,A) \to S_{\mathrm{p}}(X,A_{s_i})$
	$\pi : S_{\mathrm{p}}(X,A_{s_1}) \times \cdots \times S_{\mathrm{p}}(X,A_{s_k}) \to O_{\mathrm{p}}(X,A)$

Recall from Fig. 4 the algebras used in the construction of the algebra $\mathscr{O}$.

We will define an m-ary operation on partial spatial objects,

$$F : O_{\mathrm{p}}(X,A)^m \to O_{\mathrm{p}}(X,A),$$

as the interpretation $[\![t]\!]_{\mathscr{O}}$ of a term t over the algebra $\mathscr{O}$.

Terms over $\mathscr{O}$ have the following form. From the types of the symbols in $\mathscr{O}$ it is clear that the outermost symbol of a term $t = t(X_1,\ldots,X_k)$ of type partial spatial object must be π. Indeed, we have

$$t = \pi(t_1,\ldots,t_k),$$

where $t_1,\ldots,t_k$ are terms over $\mathscr{O}$ of type partial scalar field and k is the number of attributes. (If $k = 1$, then π is a type conversion map.)

In particular, each subterm t_i over $\mathscr{O}$ of type partial scalar field has the form

$$s_i(\pi_{g(i,1)}X_{h(i,1)},\ldots,\pi_{g(i,k_i)}X_{h(i,k_i)}),$$

where s_i is a term over $\mathscr{S}$ of type partial scalar field, and where g is a function selecting the proper projection and h is a function selecting the proper spatial object to use.

The general form of an m-ary operation on partial spatial objects with k attributes derived from the operations on partial scalar fields together with the projection and

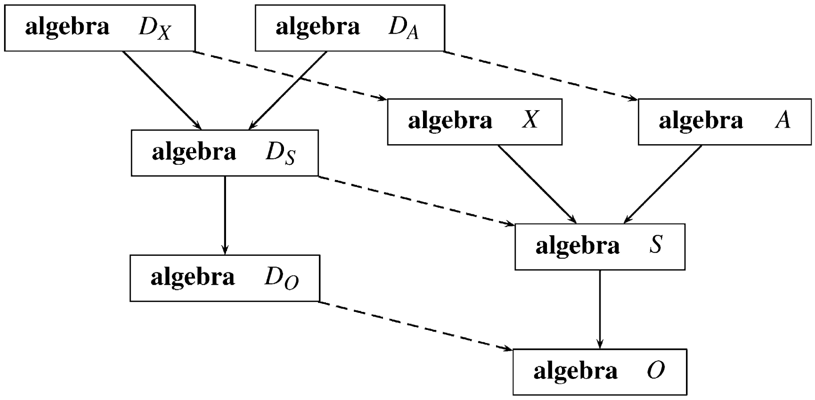

Fig. 5. Domain representations of spatial object algebra.

construction operations is

$$F(\mathbf{o}_1,\ldots,\mathbf{o}_m) = [\![t]\!]_{\mathcal{O}}(\mathbf{o}_1,\ldots,\mathbf{o}_m).$$

5.2. Representations of algebras over spatial objects

We will now describe how to create a domain representation for an algebra of partial spatial objects from domain representations of its component algebras, see Fig. 5.

5.2.1. Representing space and data algebras

In all our examples, the space and data algebras are regular Hausdorff spaces (indeed, they are all metric spaces). This is not an accident, since group structures underly many models of space. Topological groups are always regular and commonly Hausdorff (and, indeed, metrizable by the Birkhoff–Kakutani Theorem). See, for example, [1, Chapter 1].

Therefore, since the existence of upwards-closed retract domain representations is a necessary and sufficient condition for regular Hausdorff spaces (Theorem 3.1), the following is a starting point.

Choose upwards-closed retract domain representations of both the space X and the data algebra A, denoted by $(D_X, D_X^{\mathrm{R}}, \rho_X, \eta_X)$ and $(D_A, D_A^{\mathrm{R}}, \rho_A, \eta_A)$, respectively.

5.2.2. Representing scalar field algebras

The space $S_{\mathrm{p}}(X,A)$ of partial scalar fields consists of partial functions from the space X into data A. We make an initial assumption that $S_{\mathrm{p}}(X,A)$ consists precisely of the representable partial functions, i.e.,

$$S_{\mathrm{p}}(X,A) = \mathrm{Rep}_{\mathrm{p}}^{D_X,D_A}(X,A).$$

Therefore, the set $S_{\mathrm{p}}(X,A)$ will, of necessity, be a subset of $C_{\mathrm{p}}(X,A)$.

Recall from Section 3.2.2 that $[D_X \to D_A]$ is not a domain representation of $\mathrm{Rep_p}(X,A)$ in the strong sense of Section 3.1. So, results on domain representations, such as lifting theorems for operations do not hold for our representation of partial scalar fields. However, as we will show here, both pointwise extensions of data operations and spatial transformations have representations. Furthermore, if the representations of the underlying operations are effective, then the representations of these operations on scalar fields will also be effective.

Recall from Definition 3.11 what it means to represent an operation over spaces of partial functions.

Proposition 5.6. *Representations of space transformations on scalar fields can be constructed from representations of the space transformations in the space algebra. Effectivity is preserved by this construction.*

Proof. Let f be a representation of the transformation $t(p): X \to X$ for some parameter p. Define $\bar{f}: [D_X \to D_{A_s}] \to [D_X \to D_{A_s}]$ by

$$\bar{f}(g)(d) = g(f(d)).$$

It is routine to verify that $\bar{f}$ is continuous and that $\bar{f}$ is effective if f is effective. Clearly, $\bar{f}$ represents $t(p)$. □

Proposition 5.7. *Representations of pointwise extensions of data operations to scalar fields can be constructed from representations of the data operations. Effectivity is preserved by this construction.*

Proof. Let

$$\varphi: A_{s_1} \times \cdots \times A_{s_k} \to_{\mathrm{p}} A_s$$

be a partial data operation which is represented by

$$f: D_{A_{s_1}} \times \cdots \times D_{A_{s_k}} \to D_{A_s}.$$

Define

$$\bar{f}: [D_X \to D_{A_{s_1}}] \times \cdots \times [D_X \to D_{A_{s_k}}] \to [D_X \to D_{A_s}]$$

by

$$\bar{f}(g_1,\ldots,g_k)(d) = f(g_1(d),\ldots,g_k(d)),$$

where $d \in D_X$ and $g_i \in [D_X \to D_{A_{s_i}}]$. Clearly, $\bar{f}$ is continuous and effective in case f is effective.

It remains to show that $\bar{f}$ applied to some representations of partial scalar fields gives the appropriate result. Let $g_i: D_X \to D_{A_{s_i}}$ be a representation of $\sigma_i \in S_{\mathrm{p}}(X, A_{s_i})$. Let $d \in D_X^{\mathrm{R}}$ and let $x = \rho_X(d)$. If $\sigma_i(x)$ is defined then $g_i(d) \in D_{A_{s_i}}^{\mathrm{R}}$, and

$$\rho_{A_{s_i}}(g_i(d)) = \sigma_i(x).$$

Now, if φ is defined for $\sigma_1(x),\ldots,\sigma_k(x)$ then $f(g_1(d),\ldots,g_k(d))\in D^{\mathrm{R}}_{A_s}$. Furthermore

$$\begin{aligned}\rho_{A_s}(\bar{f}(g_1,\ldots,g_k)(d)) &= \rho_{A_s}(f(g_1(d),\ldots,g_k(d)))\\ &= \varphi(\rho_{A_{s_1}}g_1(d),\ldots,\rho_{A_{s_k}}g_k(d))\\ &= \varphi(\sigma_1\rho_X(d),\ldots,\sigma_k\rho_X(d))\\ &= \varphi(\sigma_1(x),\ldots,\sigma_k(x))\\ &= \bar{\varphi}(\sigma_1,\ldots,\sigma_k)(x).\end{aligned}$$

Hence $\bar{f}(g_1,\ldots,g_k)$ is a representation of $\bar{\varphi}(\sigma_1,\ldots,\sigma_k)$, i.e., $\bar{f}$ is a representation of $\bar{\varphi}$. □

5.3. *Representing the algebra over partial spatial objects*

Assuming that the scalar field algebra has a domain representation the construction of the domain representation of spatial objects is simply the Cartesian product of the representations of the constituting scalar fields. Thus, the representation is

$$D=\prod_{i=1}^{k}[D_X\to D_{A_{s_i}}],$$

where k is the number of attributes. The representable spatial objects is a subset of the space $O_{\mathrm{p}}(X,A)$ that depends on the representations chosen for the space and the data algebras, D_X and D_A. We denote the representable spatial objects by

$$O_{\mathrm{p}}^{D_X,D_A}(X,A)=\prod_{i=1}^{k}\mathrm{Rep}_{\mathrm{p}}^{D_X,D_{A_{s_i}}}(X,A_{s_i}).$$

The projections and the tupling operation as well as the evaluation operation are all represented by their domain theoretic counterpart. The rest of the operations over spatial object algebra are obtained as terms over the scalar field algebra. Hence, domain representations of these operations are readily provided by the corresponding term construction for the representing domains.

Theorem 5.8. *The algebra $O_{\mathrm{p}}^{D_X,D_A}(X,A)$ is effectively represented by D if D_X and D_A are effective representations of the space and data algebras* (*including operations*).

Proof. The domain D is an effective domain if the component domains D_X and D_A are effective. Projection, tupling and evaluation are effective domain operations. Spatial transformation and pointwise extensions of data operations are effective by Propositions 5.6 and 5.7. The compositions of effective operations used in the term constructions are again effective. □

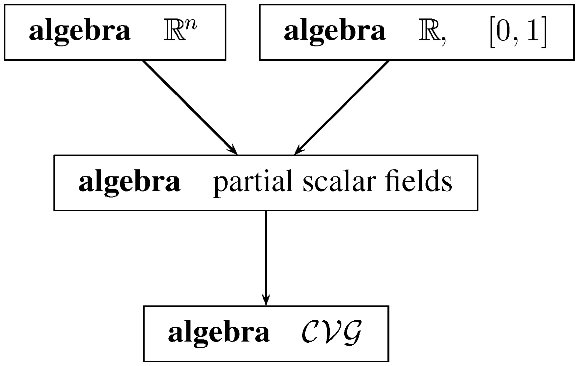

Fig. 6. CVG algebra.

6. Constructive volume geometry (CVG)

Constructive volume geometry (CVG) is a new approach to volume graphics based on high level operations that can be used to put together volume objects to form complex volume scenes. Specifically, CVG is based on making algebras of volume objects (see Fig. 6.). The volume objects are spatial objects where, essentially,

$$X = \mathbb{R}^n \quad \text{and} \quad A = ([0,1], \mathbb{R}, \ldots, \mathbb{R}).$$

Thus, given the work in Section 5, we can formally define a CVG algebra.

Definition 6.1. *CVG algebras* are spatial object algebras of the form

$$O_{\mathrm{p}}(\mathbb{R}^n, ([0,1], \mathbb{R}, \ldots, \mathbb{R})).$$

In fact, most often $n=3$, though the case $n=4$ is related to animation in volume graphics, where $\mathbb{R}^4$ is a model of the space-time continuum. See Kaufman, Cohen and Yagel [21] and Chen et al. [11] for the volume graphics paradigm, and see Chen and Tucker [12, 13] for CVG approach to volume graphics.

6.1. The 4-channel model

One basic model for volume graphics is the *4-channel model*. Here, an object is assumed to have 4 attributes (channels). The first is the opacity of the object, often called the α-channel. The other three attributes are the red, green and blue channels.

The attributes are defined by partial $\mathbb{R}$-scalar fields on $\mathbb{R}^n$. One can model the opacity by a $[0,1]$-scalar field, but for simplicity we use the real line for all attributes. The carrier for the 4-channel algebra of spatial objects is

$$O_{\mathrm{p}}(\mathbb{R}^n, A) = \prod_{i=1}^{4} S_{\mathrm{p}}(\mathbb{R}^n, \mathbb{R}).$$

Some algebras for the 4-channel model are constructed in [12]. The algebra here is based on one of theirs. However, we have added some operations to volume objects that they have included only implicitly in their algebra. We have also switched to partial scalar fields as our aim is different.

algebra	4-channel	
import	scalar fields	
carriers	$O = O_{\mathrm{p}}(\mathbb{R}^n, (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R}))$	
operations	$\mathrm{eval} : O \times \mathbb{R}^n \to \mathbb{R}^4$	
	$\pi_i : O \to S_{\mathrm{p}}(\mathbb{R}^n, \mathbb{R})$	$\pi : \prod_{i=1}^{4} S_{\mathrm{p}}(\mathbb{R}^n, \mathbb{R}) \to O$
	$\boxed{\cup} : O^2 \to O$	scale : $P \to O \to O$
	$\boxed{\cap} : O^2 \to O$	translate : $P \to O \to O$
	$\boxed{-} : O^2 \to O$	rotate : $P \to O \to O$

The space P is assumed to contain parameters suitable for the operations. For example, translate takes a point in $\mathbb{R}^n$ and moves the origin of the spatial object to that point.

Various scenes are composed using only the operations on objects described above. For example, taking the "union", $\boxed{\cup}$, of two objects creates an integrated scene with those two objects. See [13] for many illustrations.

We will show that the 4-channel model has an effective representation.

6.1.1. Representing 4-channel space $\mathbb{R}^n$

We give here a representation of the following algebra:

algebra	$\mathbb{R}^n$
import	$\mathbb{R}$
carriers	$\mathbb{R}^n$, P
operations	$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
	scale : $P \to \mathbb{R}^n \to \mathbb{R}^n$
	translate : $P \to \mathbb{R}^n \to \mathbb{R}^n$
	rotate : $P \to \mathbb{R}^n \to \mathbb{R}^n$

Let $\mathscr{R}$ be the interval domain representing the real line (see Examples 3.3 and 4.2). The representation for the Euclidean space $\mathbb{R}^n$ is obtained as the Cartesian product $\mathscr{R}^n$. The compact elements of $\mathscr{R}^n$ correspond to tuples

$$([a_1, b_1], \ldots, [a_n, b_n])$$

of rational intervals. Thus, the compact elements of the domain are always rectangular blocks.

Since all operations in the algebra are continuous, by Theorem 3.2 each operation has a domain representation. As an example, a representation of the metric is given. The representation has the virtue of being effective.

algebra	$\mathbb{R}$	
carriers	$\mathbb{R}, \mathbb{B}$	
constants	$0, 1: \mathbb{R}$	true, false: $\mathbb{B}$
operations	bound: $\mathbb{R} \to [0,1]$	max: $\mathbb{R}^2 \to \mathbb{R}$
	$+: \mathbb{R}^2 \to \mathbb{R}$	min: $\mathbb{R}^2 \to \mathbb{R}$
	$\cdot: \mathbb{R}^2 \to \mathbb{R}$	combine: $([0,1] \times \mathbb{R})^2 \to \mathbb{R}$
	$-: \mathbb{R} \to \mathbb{R}$	select: $([0,1] \times \mathbb{R})^2 \to \mathbb{R}$
	$\leq_{\mathrm{p}}: \mathbb{R}^2 \to \mathbb{B}$	cap: $([0,1] \times \mathbb{R})^2 \to \mathbb{R}$

Fig. 7. A data algebra for 4-channel.

To find a function representing the metric on $\mathbb{R}^n$ we define $\bar{d}: \mathscr{R}_c^n \to \mathscr{R}$ by

$$\bar{d}(([a_1,b_1],\ldots,[a_n,b_n]),([c_1,d_1],\ldots,[c_n,d_n])) \\ = \{[e,f] \in P\colon\ e \leqslant \sqrt{\mu_1^2+\cdots+\mu_n^2} \text{ and } f \geqslant \sqrt{v_1^2+\cdots+v_n^2}\},$$

where

$$\mu_i = \min\{|x-y|\colon\ x \in [a_i,b_i], y \in [c_i,d_i]\}$$

and

$$v_i = \max\{|x-y|\colon\ x \in [a_i,b_i], y \in [c_i,d_i]\}.$$

The function $\bar{d}$ is monotone and hence extends uniquely to a continuous function $\bar{d}: \mathscr{R}^n \to \mathscr{R}$. Clearly, $\bar{d}$ represents the metric d.

Lemma 6.2. *The algebra* $\mathbb{R}^n$ *has an effective upwards-closed retract domain representation.*

Proof. The Cartesian product $\mathscr{R}^n$ of $\mathscr{R}$ is an upwards-closed retract representation of the space $\mathbb{R}^n$. By Theorem 3.2 each operation has a representation. We leave to the reader to verify that the operations have effective representations. □

6.1.2. Representation for 4-channel data

We give here a representation for the algebra in Fig. 7, where $\leqslant_{\mathrm{p}}$ is the partial version of the usual ordering; bound takes a real and bounds it to the interval $[0,1]$ (used to guarantee that the opacity attribute is within the unit interval); and combine, select and cap are the partial versions of the operations defined in [12] to handle the colour attributes.

The representation of the set of reals will be $\mathscr{R}$ as above. As our representation of the booleans we will choose $(\mathbb{B}_\perp, \mathbb{B}, \iota, \iota)$.

All total continuous operations of the algebras of the ordered real line and the unit interval are easily represented by domain functions. The domain functions are defined on the compact elements (the rational intervals) in the obvious way (recall that the endpoints of the intervals are always rational). The following examples show how some of the operations are given domain representations.

Example 6.3. Define $f_+ : \mathscr{R}_c^2 \to \mathscr{R}$ by

$$f_+([a,b],[c,d]) = [a+c, b+d].$$

The monotone function f_+ extends uniquely to a continuous domain function $f_+ : \mathscr{R}^2 \to \mathscr{R}$. Let $d, d' \in \mathscr{R}^{\mathrm{R}}$ represent x and y in $\mathbb{R}$ respectively. Then $f_+(d,d')$ will contain arbitrarily small intervals around the point $x+y$, i.e., $f_+(d,d')$ represents $x+y$. Hence, f_+ represents addition. Clearly, f_+ is an effective domain function.

Example 6.4. Define $g : \mathscr{R}_c \to \mathscr{R}$ by

$$g([a,b]) = [\max(a,0), \min(b,1)].$$

The monotone function g extends uniquely to a continuous function $g : \mathscr{R} \to \mathscr{R}$. Clearly, g is an effective domain function representing the bounding operation.

The ordering is a discontinuous function so we are forced to consider a partial version of it to get continuity and computability.

Example 6.5. The partial version $\leqslant_{\mathrm{p}}$ of the ordering is represented by $f : \mathscr{R}_c^2 \to \mathbb{B}_\perp$ defined as

$$f([a,b],[c,d]) = \begin{cases} \text{true} & \text{if } b < c, \\ \text{false} & \text{if } d < a, \\ \uparrow & \text{otherwise.} \end{cases}$$

Clearly, f is monotone, and hence extends uniquely to a continuous function $f : \mathscr{R}^2 \to \mathbb{B}_\perp$. The function f represents the ordering at all points except at discontinuities. Moreover, f is effective.

The remaining operations combine, select and cap can also be represented in their partial versions. Partiality is necessary because of comparisons made in the definition of the operations.

Lemma 6.6. *The data algebra of* 4-*channel has an effective upwards-closed retract domain representation.*

Proof. Take the domain $\mathscr{R}$ together with the representations of operations discussed above. □

6.1.3. Representing partial scalar fields for 4-channel

Lemma 6.7. *The 4-channel operations on partial scalar fields are domain represented by effective domain functions on* $[\mathscr{R}^n \to \mathscr{R}]$.

Proof. By Propositions 5.6 and 5.7. □

To illustrate the above lemma we look at the representation obtained for the pointwise extension of addition.

Example 6.8. We look at the representation of the addition operation on partial scalar fields in $\mathrm{Rep}_{\mathrm{p}}(\mathbb{R}^n, \mathbb{R})$. By Lemma 6.6 addition has an effective representation, call this representation f_+. The pointwise extension $\bar{+}$ of addition is represented by the domain function $\bar{f}_+ : [\mathscr{R}^n \to \mathscr{R}]^2 \to [\mathscr{R}^n \to \mathscr{R}]$. Recall from Proposition 5.7 that $\bar{f}_+$ is defined on compact elements by

$$\bar{f}_+(c, c')(d) = f_+(c(d), c'(d)),$$

where

$$c = \langle R_1; [a_1, b_1] \rangle \sqcup \cdots \sqcup \langle R_k; [a_k, b_k] \rangle$$

and

$$c' = \langle R'_1; [a'_1, b'_1] \rangle \sqcup \cdots \sqcup \langle R'_m; [a'_m, b'_m] \rangle.$$

The result of $\bar{f}_+(c, c')$ is

$$\bigsqcup \{ \langle R_i \sqcup R'_j, f_+([a_i, b_i], [a'_j, b'_j]) \rangle : \ \mathrm{Cons}(R_i, R'_j) \}.$$

The monotone function $\bar{f}_+$ extends uniquely to a continuous function representing $\bar{+}$. Moreover, $\bar{f}_+$ is an effective function.

6.1.4. Representing 4-channel algebra

Finally, we will establish a domain representation of the CVG algebra 4-channel.

The space of partial spatial objects in the CVG algebra 4-channel will be represented by the domain

$$\mathscr{V} = [\mathscr{R}^n \to \mathscr{R}]^4.$$

This is an example of a *voxel domain* defined later. Projection and tupling are represented by the corresponding operations on the domain representations. All other operations in the CVG algebra 4-channel are constructed from the operations on partial scalar fields together with projection and tupling. The following example shows how one of the operations are represented.

Example 6.9. The intersection operation $\boxed{\cap}$ of the 4-channel CVG algebra is defined in [12] by

$$\begin{aligned}(\sigma_\alpha, \sigma_\mathrm{r}, \sigma_\mathrm{g}, \sigma_\mathrm{b}) \boxed{\cap} (\tau_\alpha, \tau_\mathrm{r}, \tau_\mathrm{g}, \tau_\mathrm{b}) = (&\min(\sigma_\alpha, \tau_\alpha),\\ &\mathrm{select}(\sigma_\alpha, \sigma_\mathrm{r}, \tau_\alpha, \tau_\mathrm{r}),\\ &\mathrm{select}(\sigma_\alpha, \sigma_\mathrm{g}, \tau_\alpha, \tau_\mathrm{g}),\\ &\mathrm{select}(\sigma_\alpha, \sigma_\mathrm{b}, \tau_\alpha, \tau_\mathrm{b})),\end{aligned}$$

where min denotes the pointwise extension of the minimum operation, and select is the pointwise extension of

$$(x, s, y, t) \mapsto \begin{cases} s & \text{if } x \geqslant y, \\ t & \text{if } x < y. \end{cases}$$

By Lemma 6.7 the pointwise extensions of min and select have effective representations. Let these be represented by g and h respectively. By composition of these representations, together with the projection and construction operations, we get a representation for $\boxed{\cap}$ on the domain $\mathscr{V}$.

Define $f : \mathscr{V}^2 \to \mathscr{V}$ by

$$\begin{aligned} f(x, y) = \pi(&\bar{g}(\pi_1(x), \pi_1(y)), \\ &\bar{h}(\pi_1(x), \pi_2(x), \pi_1(y), \pi_2(y)), \\ &\bar{h}(\pi_1(x), \pi_3(x), \pi_1(y), \pi_3(y)), \\ &\bar{h}(\pi_1(x), \pi_4(x), \pi_1(y), \pi_4(y))). \end{aligned}$$

Then f is a representation of $\boxed{\cap}$. Moreover, since f is a composition of computable operations, f itself is a computable operation.

Theorem 6.10. *The CVG algebra 4-channel has an effective domain representation.*

Proof. Let $\mathscr{V}$ be the voxel domain $[\mathscr{R}^n \to \mathscr{R}]^4$. By constructions similar to the one in Example 6.9 the operations of 4-channel have effective representations on $\mathscr{V}$. $\square$

6.2. Representing a general CVG algebra

Assume that a CVG algebra is built from a space algebra $\mathbb{R}^n$ and a data algebra $A = ([0, 1], \mathbb{R}, \ldots, \mathbb{R})$. The space and data sets can be domain represented by $\mathscr{R}^n$ and $\mathscr{R}$, respectively.

The representations of the partial scalar fields from $\mathbb{R}^n$ into the data algebra will be the domains $[\mathscr{R}^n \to \mathscr{R}]$. The set $\mathrm{Rep}_{\mathrm{p}}(\mathbb{R}^n, \mathbb{R})$ is the set of partial scalar fields which have representations in the domain $[\mathscr{R}^n \to \mathscr{R}]$.

The compact elements of the domain $[\mathscr{R}^n \to \mathscr{R}]$ are objects of the following kind:

$$\langle R_1; [a_1, b_1] \rangle \sqcup \cdots \sqcup \langle R_k; [a_k, b_k] \rangle,$$

where R_i represents some rectangular block region in the space $\mathbb{R}^n$ of the form $R_i = ([x_1, x_1'], \ldots, [x_n, x_n'])$, where $x_j, x_j' \in \mathbb{Q} \cup \{-\infty, \infty\}$, and $[a_i, b_i]$, where $a_i, b_i \in \mathbb{Q} \cup \{-\infty, \infty\}$, is a finite approximation of the value of the scalar field within that region.

There is no requirement of uniformity between the regions with regard to size and shape (except that they are all rectangular blocks). A perfectly allowable region is, for example, the whole space. Clearly, we do not require that $\bigcup R_i$ covers the space $\mathbb{R}^n$.

Each of the approximations $\langle R_i; [a_i, b_i]\rangle$ of a partial scalar field is very similar to our informal notion of voxel (defined in the Introduction). In fact, in the context of volume graphics, they may be said to be *single attribute voxels*, since they give an approximation for only one of the attributes. Shortly, we will build formal voxels from these (Definition 6.12).

The following theorem answers the question: what is sufficient for a CVG algebra to have an effective domain representation?

Theorem 6.11. *A CVG algebra has an effective domain representation if the spatial transformations and the data operation are effectively representable on the interval domain representation of the reals.*

Proof. The operations on partial scalar fields are spatial transformations, pointwise extensions of data operations, and evaluation. By Propositions 5.6 and 5.7 these operations have effective representations.

Tupling, projection, and evaluation are effectively represented on the domains. Remaining operations on the partial spatial objects are constructed from the above operations by composition and are therefore also effectively representable. □

6.3. The voxel notion in CVG

We will here formalise the notion of voxel and investigate its connection with the domain representations of CVG algebras. Of special interest is the fact that certain standard concepts of domain representability are closely related to the notion of voxel in volume graphics, and that domain models therefore offer a viable solution to the general semantic modelling problem mentioned in the introduction for volume graphics and CVG.

The domain representation of the space of partial volume objects of a general CVG algebra is

$$\mathscr{V} = [\mathscr{R}^n \to \mathscr{R}]^k.$$

The compact elements of $\mathscr{V}$ approximating a partial CVG object are of the form

$$(c_1, \ldots, c_k),$$

where c_i is a compact element approximating the ith scalar field, i.e.,

$$c_i = \langle R_{i1}; [a_{i1}, b_{i1}]\rangle \sqcup \cdots \sqcup \langle R_{im_i}; [a_{im_i}, b_{im_i}]\rangle.$$

We now give our formal definition of the notion of voxel.

Definition 6.12. An n-dimensional *voxel* with k attributes is a pair

$$(R, ([a_1, b_1], \ldots, [a_k, b_k])),$$

where R is a rectangular block region of space $\mathbb{R}^n$ with rational endpoints, and $a_i, b_i \in \mathbb{Q}$.

The notion of approximation that we use is that every possible value should be included in the approximation, so a typical approximation for a real valued attribute will be an interval bounding the possible values.

Now, a voxel $v=(R,([a_1,b_1],\dots,[a_k,b_k]))$ corresponds to the compact element

$$c=(\langle R;[a_1,b_1]\rangle,\dots,\langle R;[a_k,b_k]\rangle).$$

Lemma 6.13. *The cusl of compact elements of $\mathscr{V}$ is generated by the voxels from the operation $\sqcup$.*

Proof. Any compact element in the domain representation of CVG may be construed as a finite supremum of voxels as follows. Take a general compact element $c=(c_1,\dots,c_k)$, where

$$c_i=\langle R_{i1};[a_{i1},b_{i1}]\rangle\sqcup\cdots\sqcup\langle R_{im_i};[a_{im_i},b_{im_i}]\rangle.$$

The compact element c can be written as

$$c=(c_1,\dots,c_k)=\bigsqcup_{i,j}(\langle\bot;\bot\rangle,\dots,\langle\bot;\bot\rangle,\langle R_{ij};[a_{ij},b_{ij}]\rangle,\langle\bot;\bot\rangle,\dots,\langle\bot;\bot\rangle),$$

which corresponds to the following supremum of voxels:

$$\bigsqcup_{i,j}(R_{ij},([-\infty,\infty],\dots,[-\infty,\infty],[a_{ij},b_{ij}],[-\infty,\infty],\dots,[-\infty,\infty])). \qquad \square$$

The motivation for the following definition is that the domain is uniquely determined by the voxels (via completion of the cusl).

Definition 6.14. The domain representation of a space of CVG objects

$$\mathscr{V}=[\mathscr{R}^n\to\mathscr{R}]^k$$

is called the *voxel domain*.

We claim that the voxel domain contains all "computable" partial volume objects of CVG, and that the partial volume objects, i.e., ideal elements of the voxel domain, are generated from voxels.

We now give two examples of voxel constructions from the opacity only model $O_{\mathrm{p}}(\mathbb{R}^n,[0,1])$ of CVG.

Example 6.15. This example shows how to approximate the opaque unit disc, radius 1 and centre in the origin, in the opacity only version of CVG. This example uses $\mathbb{R}^2$ as space since it makes the pictures easier to understand.

Fig. 8a depicts the disc as a volume object. Figs. 8b–d are approximations of this disc built from voxels. The value of the opacity attribute is 1 for opaque points and 0 for transparent points. In the picture, opaque is depicted by black and transparent

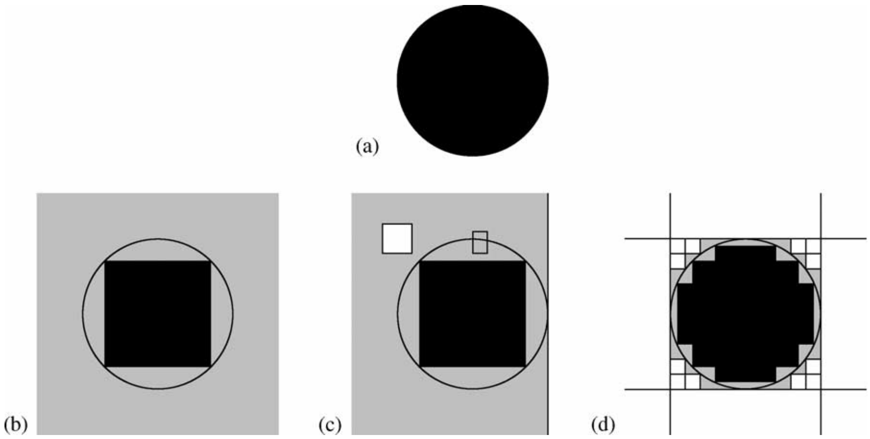

Fig. 8. A disc in $\mathbb{R}^2$ and approximations of it.

by white. Gray in the approximations depicts a point where there is no information whether it is opaque or not.

The approximation of Fig. 8b is the single voxel (compact element of $\mathscr{V} = [\mathscr{R}^2 \to \mathscr{R}]$)

$$\langle [-0.7, 0.7] \times [-0.7, 0.7];\ [1, 1] \rangle$$

approximating the opaque disc from inside.

The approximation of Fig. 8c consists of four voxels. The black square approximation of the disc from inside, the white square approximation from the outside, the gray (no information) approximation over the boundary, and the white half-plane. The approximation is the following supremum of voxels:

$$\langle [-0.7, 0.7] \times [-0.7, 0.7];\ [1, 1] \rangle \sqcup \langle [-1.2, 0.8] \times [-0.8, 1.2];\ [0, 0] \rangle$$
$$\sqcup \langle [0, 0.2] \times [0.8, 1.1];\ [0, 1] \rangle \sqcup \langle [1.01, \infty] \times [-\infty, \infty];\ [0, 0] \rangle.$$

The approximation of Fig. 8d is built from three rectangular approximations of the inside, four half-plane approximations from outside, and eight rectangular approximations from outside. Written as a supremum of voxels:

$$\langle [-0.7, 0.7] \times [-0.7, 0.7];\ [1, 1] \rangle \sqcup \langle [-0.9, 0.9] \times [-0.4, 0.4];\ [1, 1] \rangle$$
$$\sqcup \langle [-0.4, 0.4] \times [-0.9, 0.9];\ [1, 1] \rangle \sqcup \langle [1.03, \infty] \times [-\infty, \infty];\ [0, 0] \rangle$$
$$\sqcup \langle [-\infty, \infty] \times [1.03, \infty];\ [0, 0] \rangle \sqcup \langle [-\infty, -1.03] \times [-\infty, \infty];\ [0, 0] \rangle$$
$$\sqcup \langle [-\infty, \infty] \times [-\infty, -1.03];\ [0, 0] \rangle \sqcup \langle [0.61, 1] \times [0.8, 1];\ [0, 0] \rangle$$
$$\sqcup \langle [0.8, 1] \times [0.61, 1];\ [0, 0] \rangle \sqcup \langle [-1, -0.61] \times [0.8, 1];\ [0, 0] \rangle$$
$$\sqcup \langle [-1, -0.8] \times [0.61, 1];\ [0, 0] \rangle \sqcup \langle [0.61, 1] \times [-1, -0.8];\ [0, 0] \rangle$$
$$\sqcup \langle [0.8, 1] \times [-1, -0.61];\ [0, 0] \rangle \sqcup \langle [-1, -0.61] \times [-1, -0.8];\ [0, 0] \rangle$$
$$\sqcup \langle [-1, -0.8] \times [-1, -0.61];\ [0, 0] \rangle.$$

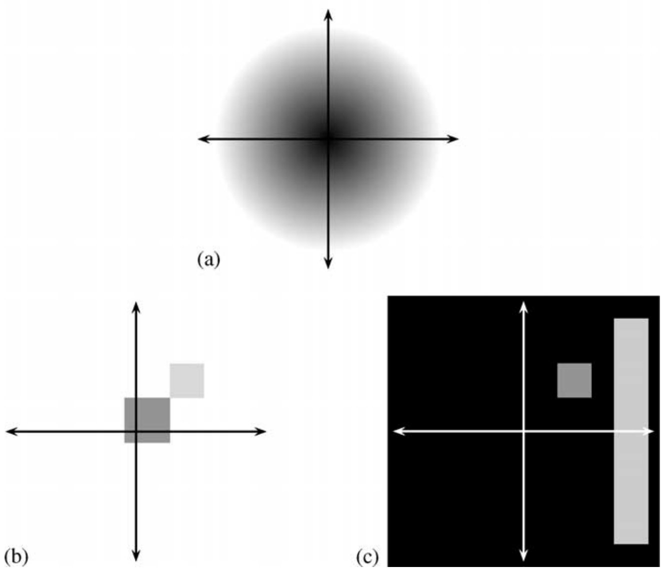

Fig. 9. A continuously varying opacity volume object.

Example 6.16. This example shows how to approximate a continuously varying opacity field. Again, this example uses $\mathbb{R}^2$ as space.

Fig. 9a shows the volume object. The gray-scale is now used to depict intermediate opacity. A picture of the following approximation of the volume object is given in Fig. 9b and c:

$$\langle[-0.1, 0.3] \times [-0.1, 0.3];\ [0.42, 1]\rangle \sqcup \langle[0.8, 1.1] \times [-1, 1];\ [0, 0.2]\rangle \sqcup \langle[0.3, 0.6] \times [0.3, 0.6];\ [0.15, 0.42]\rangle.$$

Fig. 9b shows the lower bound of the opacity determined by the approximation, and Fig. 9c shows the upper bound of the opacity determined by the same approximation. Both pictures are needed in order to understand an approximation since any approximation gives only an interval of possible values for each point in the space. The first voxel corresponds to the square containing the origin. This square cannot be seen in the right picture since the approximation allows the volume object to be completely opaque within the square. The second voxel is the rectangle. Similarly, this cannot be seen in the left picture. The third voxel is the smaller square. The area not covered by the voxels may obviously attain any value from 0 (white) to 1 (black).

6.4. CSG and CVG

The fundamental objects of *constructive solid geometry* (CSG) are usually given as subsets of euclidean space. The simplest way to incorporate CSG into our framework is to change the viewpoint of solids to be functions from space into the booleans. With this modification CSG has a domain representation similar to the one for CVG.

However, CSG may also be embedded as a subalgebra of CVG. Take the opacity-only model of CVG. Then solids are characterised as those partial volume objects taking only the values 0 and 1. We note that the operations of $\boxed{\cup}$ and $\boxed{\cap}$ correspond to union and intersection of solids, and that the $\boxed{-}$ operation applied to the everywhere opaque volume object and a solid will yield the complement of the solid. Summarising we have the following proposition.

Proposition 6.17. *constructive solid geometry is embedded into the opacity-only model of constructive volume geometry.*

The above representation of CSG is effective, cf. [17, 18] for another approach to introduce effectivity to CSG. Alternate approaches are possible based on semicomputable subsets of $\mathbb{R}^n$, see [10].

7. Concluding remarks

We have investigated the domain representation of partial topological algebras of partial continuous functions between regular Hausdorff spaces. We have looked at the special case where these spaces are subspaces of $\mathbb{R}^n$ and the algebras are used for constructive volume geometry. The theoretical problem of classifying the partial functions that have liftings is both interesting and important, and much remains to be done.

The application of domain representations to volume graphics provides an elegant solution to the problem of finding a semantic framework for volume graphics. The domain concepts seem ideal for capturing the process by which voxels approximate volume objects. The domain model of volume graphics can also act as a semantic model for programming constructs and as correctness criteria for volume graphics techniques. The model also provides a new application for exact real arithmetic.

For many problems of scientific simulation—from the EMR of aerofoils to the electric activity of cardiac tissue—the states of systems are spatial objects and operations are needed to describe the dynamic behaviour of the system. Thus, there is scope for further investigations of topological algebras of spatial objects in several areas of physical modelling, as well as visualisation and graphics. There is a common algebraic structure to many dynamical systems, both continuous and discrete, that enables a general theory of physical and computing systems to be attempted, see [25]. This structure also enables us to study the dynamical systems using the methods of this paper.

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