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Reducibility of domain representations and Cantor–Weihrauch domain representations

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We introduce a notion of reducibility of representations of topological spaces and study some basic properties of this notion for domain representations.
A representation reduces to another if its representing map factors through the other representation. Reductions form a pre-order on representations. A spectrum is a class of representations divided by the equivalence relation induced by reductions. We establish some basic properties of spectra, such as, non-triviality. Equivalent representations represent the same set of functions on the represented space.
Within a class of representations, a representation is universal if all representations in the class reduce to it. We show that notions of admissibility, considered both for domains and within Weihrauch's TTE, are universality concepts in the appropriate spectra. Viewing TTE representations as domain representations, the reduction notion here is a natural generalisation of the one from TTE.
To illustrate the framework, we consider some domain representations of real numbers and show that the usual interval domain representation, which is universal among dense representations, does not reduce to various Cantor domain representations. On the other hand, however, we show that a substructure of the interval domain more suitable for efficient computation of operations is equivalent to the usual interval domain with respect to reducibility.

1. Introduction

A standard method of computing on a set X of data is to make a representation R of the data and to compute on R. Such methods have been called concrete computability theories in Tucker and Zucker (2000; 2004). The following question arises immediately: to what extent does computability on X depend on the *choice* of R? For any concrete computability there is the problem of clarifying the applicability of representations for different tasks.

One approach to resolving this problem is to look at the intrinsic properties of representations, as was done in, for example, Blanck (2000). As an alternative approach, we present here a framework for relating representations to each other, which allows us to study their relative merits. This is done by studying *reductions* (or *translations*) between representations. This is parallel to the use of reductions for *numberings* (as used in Computable Algebra). Reducibility between numberings (when one numbering factors through another) is one of the basic tools in studying numberings (Mal'cev 1961; Ershov 1973; 1975; 1977a; Stoltenberg-Hansen and Tucker 1995; 1999a). We will generalise reducibility to a very general class of representations of topological spaces and study basic properties of reducibility, in particular, for domain representations.

Our aim is to study computability on uncountable structures (usually topological spaces). A simple numbering is not possible for an uncountable structure. We therefore have to rely on computations on some numbered set of approximations. For example, real number computations can be performed using the countable set of rational intervals as approximations. A general method for giving computability theory to a large class of topological spaces is to use *domain representations.*

Domain representablility is general (Blanck 2000) in that any T_0 space can be represented. However, not all of these representations are equally suitable for performing computations. It is desirable that the operation under consideration should be represented by effective functions, but the effectivity of an operation may depend on the choice of representation. However, there are some sufficient conditions for representations of continuous functions to exist, for example, with a *dense retract representation* all continuous functions are representable. Such representations allow us to study topological algebras.

For a class R of representations of a space X and a class Φ of functions between representations, we introduce our notion of Φ-*reducibility* for representations. Reducibility introduces a pre-order on R and thereby an equivalence relation, $\equiv_\Phi$. A Φ-*spectrum* is the quotient of the class R of representations divided by the equivalence relation $\equiv_\Phi$. Limiting the class of functions available for reductions will give finer-grained spectra. For example, the spectra arising from effective reductions are finer than the spectra arising from continuous reductions.

We give examples showing that even the coarse-grained spectrum of continuous reducibility over all domain representations of a topological space X is non-trivial unless the topology of X is very coarse or very fine.

Some intrinsic properties of domain representations interact with our theory of reductions, so, for example, retract representations reduce continuously only to retract representations.

The importance of density in domain representations has an information theoretic explanation in that non-dense representations contain non-consistent information or ‘garbage’. When we restrict our attention to dense representations and continuous reductions, there is a top element in the spectrum of all dense domain representations, namely the equivalence class of dense retract representations.

If there exists a top element in a spectrum, we call a representation belonging to it *universal* as all other representations in that spectrum will reduce to it. We show that notions of admissibility as studied in, for example, Schröder (2002) and Hamrin (2005), are in fact notions of universality in the appropriate spectrum. Among the class of representations considered, universal representations most closely capture the structure of the represented space.

To illustrate the framework, we conclude by studying some representations of real numbers. The usual interval domain representation of the reals is known to be universal among countably based dense representations (ω-admissible). We show that a particular substructure of the interval domain, where operations on exact reals can be more efficiently computed, is continuously equivalent to the interval domain (although it is a bifinite domain rather than a Scott–Ershov domain), and hence that it can be used interchangeably. We compare some Cantor domain representations and show how these relate to the interval domain representation. We also give a signed binary domain representation of the reals and show that it is effectively equivalent to the interval domain representation.

1.1. *Related work*

The use of ordered structures to study effectivity on uncountable structures has appeared from time to time, for example, in higher types (Ershov 1977b) and weighted cpos (Weihrauch and Schreiber 1981). Weihrauch developed the idea of representations further in his successful Type-2 Theory of Effectivity (TTE), where he uses representations from Baire or Cantor space (Weihrauch 1987; 2000). Both of these spaces have natural embeddings into domains and can in this sense be viewed as a restrictive version of domain representations.

A more systematic study of representations from ordered structures, as well as the notion of *domain representation*, originate with Stoltenberg-Hansen and Tucker (1985; 1988; 1991; 1993; 1995; 2007). Continuous domain representations are used in Edalat (1995a; 1995b; 1997), Edalat and Heckmann (1998) and Edalat and Lieutier (1999; 2002) to cover application areas from integration and differential equations to solid geometry. For further papers on algebraic domain representations see Berger (1996; 1999), Blanck (1995; 1997a; 1997b; 1999; 2000), di Gianantonio(1996), Hamrin (2005), Normann (1996) and Waagbø (1997).

Weihrauch also introduced reduction notions (continuous and effective) between the representations used in TTE. Here, we extend the reducibility notion from TTE to domain representations and note that TTE representations are a special case of domain representations. Thus, we can directly relate TTE representations and domain representations within our framework. Weihrauch's representation theory has been an inspiration for this work.

Domain representations have been reinvented with the introduction of *equilogical spaces* (Scott 1996). The category **Equ** of equilogical spaces was shown in Bauer *et al.* (2004) to be equivalent to the category **DPER**(**Dom**) of dense partial equivalence relations over domains, which is just another view of dense domain representations. We work here in the setting of **PER**(**Dom**), which is also a cartesian closed category containing **DPER**(**Dom**) as a sub-ccc. See also Birkedal *et al.* (1998), Rosolini (2000), Bauer (2002), Menni and Simpson (2002), Bauer *et al.* (2004) and Lietz (2004).

2. Preliminaries

2.1. *Domains*

We will briefly give some background to domain theory. For a fuller background on domains, see Abramsky and Jung (1994) and Stoltenberg-Hansen *et al.* (1994).

Let $D = (D, \sqsubseteq)$ be a partially ordered set. A subset $A \subseteq D$ is an *upper set* (dual *lower set*) if $x \in A$ and $x \sqsubseteq y$ implies $y \in A$. Let $\uparrow A = \{y \in D : \exists x \in A(x \sqsubseteq y)\}$. We will abbreviate $\uparrow\{x\}$ by $\uparrow x$. A subset $A \subseteq D$ is *directed* if $A \neq \varnothing$ and whenever $x, y \in A$, there is $z \in A$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$. The supremum, or least upper bound, of A (if it exists) is denoted by $\bigsqcup A$.

A *complete partial order*, abbreviated *cpo*, is a partial order $D = (D; \sqsubseteq, \bot)$ such that $\bot$ is the least element in D and any directed set $A \subseteq D$ has a supremum $\bigsqcup A$.

Let D be a cpo. An element $c \in D$ is *compact* if for each directed $A \subseteq D$,

$$c \sqsubseteq \bigsqcup A \implies (\exists a \in A)(c \sqsubseteq a).$$

The set of compact elements of D is denoted by D_{c}. A domain D is *algebraic* if for all $x \in D$, $\mathrm{approx}(x) = \{a \in D_{\mathrm{c}} : a \sqsubseteq x\}$ is directed and $\bigsqcup \mathrm{approx}(x) = x$.

A cpo D is *consistently complete* if $\bigsqcup A$ exists in D whenever $A \subseteq D$ is a consistent set, that is, has an upper bound.

Definition 2.1. A *Scott–Ershov domain*, or simply *domain*, is a consistently complete algebraic cpo.

The topology normally used on domains is called the Scott topology. Let D be an algebraic cpo. A subset U of D is open if:

(i) U is an upper set; and
(ii) $x \in U$ implies that there exists $a \in \mathrm{approx}(x)$ such that $a \in U$.

An easy observation is that the Scott topology on a domain is T_0. However, the Scott topology fails to be T_1 on all domains except the trivial domain consisting of a single element.

The sets $\uparrow a$, for $a \in D_{\mathrm{c}}$, constitute a base for the Scott topology on a domain D.

Let D and E be domains. A function $f : D \to E$ is Scott continuous if f is monotone and

$$f(\bigsqcup A) = \bigsqcup f[A],$$

for any directed $A \subseteq D$. The notion of Scott continuity coincides with the notion of continuity induced from the Scott topology on the domains.

Any continuous function between domains is determined by its values on the compact elements. Let D be an algebraic cpo and E be a cpo, and let $f : D_{\mathrm{c}} \to E$ be a monotone function. Then there exists a unique continuous extension $g : D \to E$ of f such that $f = g|_{D_{\mathrm{c}}}$.

Domains are often constructed as the completion of some underlying structure. We will now describe the type of structure from which Scott–Ershov domains are constructed.

The compact elements D_{c} of a Scott–Ershov domain D form a conditional upper semilattice with least element, abbreviated *cusl*. That is, a cusl is a partially ordered set where a least upper bound exists for every pair of elements that have an upper bound.

An *ideal* is a directed lower set. The ideal completion over a cusl P is the set of all ideals over P, denoted $\mathrm{Idl}(P)$. When ordered by set inclusion, the ideal completion of a cusl forms a Scott–Ershov domain. For a in a cusl P, $\downarrow a$ is an ideal, the *principal ideal* generated by a. The compact elements of $\mathrm{Idl}(P)$ are the principal ideals $\downarrow a$, for $a \in P$.

The representation theorem for Scott–Ershov domains tells us that any Scott–Ershov domain is the ideal completion of a cusl.

Theorem 2.2. Let D be a Scott–Ershov domain. Then $\mathrm{Idl}(D_{\mathrm{c}}) \cong D$.

We clearly have the following equivalence: for $I \in \mathrm{Idl}(P)$,

$$\downarrow a \subseteq I \iff a \in I.$$

Thus the sets $B_a = \{I \in \mathrm{Idl}(P) : a \in I\}$ for $a \in P$ form a base for the Scott topology on $\mathrm{Idl}(P)$.

Definition 2.3. A domain D is *effective* if there exists a numbering $\alpha : \Omega_\alpha \to D_{\mathrm{c}}$, where $\Omega_\alpha \subseteq \mathbb{N}$, making the structure $(D_{\mathrm{c}}, \sqsubseteq, \mathrm{Cons}, \sqcup, \perp)$ computable.

2.2. *Representations*

We will now give some background on representations of spaces. We will give a more general setting than the domain representations considered in Blanck (2000), but we still aim for representations using some type of domain. We will adjust the terminology to cope with the more general framework.

Definition 2.4.

(i) A *weak representation* of a topological space X is a triple $(D, D^{\mathrm{R}}, \rho)$, where D is a topological space, $D^{\mathrm{R}} \subseteq D$ with the subspace topology and $\rho : D^{\mathrm{R}} \to X$ is continuous and onto.

(ii) A *quotient representation* is a weak representation where ρ is a quotient map.

The word representation will be used without qualification to mean a weak representation.

This notion of representation was studied in Blanck (2000), but there D was required to be a domain. We will always have domain representations in mind, but define the notion as generally as possible.

When needed, we write *continuous cpo representation, domain representation*, and so on, to specify the kind of space that D is. We will primarily focus on Scott–Ershov domains and algebraic cpos, since, by Proposition 2.10, any continuous cpo representation can be used to construct an algebraic cpo representation without losing any property considered herein.

The notion of representations introduced above covers all *naming systems* used in TTE, that is, both *notations* and *representations.* In fact, all TTE naming systems can be constructed as Scott–Ershov domain representations using simple and specific domains.

The set D^{R} above will be called the set of *representing elements.* For a domain-like structure D, the set D^{R} is also known as a *totality* on D. If D is a domain, the ordering of the domain can be interpreted as an information ordering. With this interpretation, the domain contains both proper approximations and total or complete representations of elements of X, the latter constituting the set D^{R}. Intuitively, D^{R} consists of those domain elements that contain sufficient information to completely determine an element in X *via* ρ.

Definition 2.5. An *effective domain representation* is a domain representation $(D, D^{\mathrm{R}}, \rho)$ where the domain D is effective.

Going beyond the type of space D used in a representation, we make use of the following important characteristics of representations.

Definition 2.6.

(i) A representation $(D, D^{\mathrm{R}}, \rho)$ is *dense* if D^{R} is dense in D.
(ii) A *retract representation* of X is a quadruple $(D, D^{\mathrm{R}}, \rho, \eta)$ where $(D, D^{\mathrm{R}}, \rho)$ is a representation, and $\eta : X \to D^{\mathrm{R}}$ is a continuous function such that $\rho\eta = \mathrm{id}_X$.

For a retract representation $(D, D^{\mathrm{R}}, \rho, \eta)$, we have that ρ is a quotient and that $\eta\rho$ is a retraction on D^{R}. In fact, X will be homeomorphic to the retract of D^{R}. In a retract representation, a canonical representative can be found continuously from any representation of an element of X.

Definition 2.7. Let $(D, D^{\mathrm{R}}, \rho_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ be representations of X and Y, respectively. A function $f : X \to Y$ is *represented* by a continuous function $\bar{f} : D \to E$ if $\rho_E \bar{f}(x) = f\rho_D(x)$ for all $x \in D^{\mathrm{R}}$ (in particular, $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$).

$$\begin{tikzcd}
D \arrow[r, "\bar{f}"] & E \\
D^{\mathrm{R}} \arrow[u, hook, "\iota"] \arrow[r, "\bar{f}"] \arrow[d, "\rho_D"'] & E^{\mathrm{R}} \arrow[u, hook, "\iota"'] \arrow[d, "\rho_E"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The functions between the subsets of representing elements are restrictions of functions. To avoid a clumsy explicit restriction notation, as in $\bar{f}|_{D^{\mathrm{R}}} : D^{\mathrm{R}} \to E^{\mathrm{R}}$, we write $\bar{f} : D^{\mathrm{R}} \to E^{\mathrm{R}}$ and trust the reader to understand this as the restriction to the indicated domain of the function.

Let $(D, D^{\mathrm{R}}, \rho_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ be representations of X and Y, respectively, and let $\bar{f} : D \to E$ be continuous such that $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$. If $\bar{f}$ respects the equivalence relations induced by ρ_D and ρ_E, then $\bar{f}$ represents a well-defined function $f : X \to Y$. Furthermore, if ρ_D is a quotient map, then f is continuous, since then the topology of X is fine enough.

For a topological space X, we define the following classes of representations of X.

Definition 2.8. For a topological space X, let $\mathbf{Rep}(X)$ denote the class of all representations of X and $\mathbf{DRep}(X)$ denote the class of all domain representations $(D, D^{\mathrm{R}}, \rho)$ of X.

We also consider subclasses of these classes of representations. In particular, subclasses of dense representations and subclasses of representations of bounded size.

Definition 2.9. Let $\mathbf{R}(X)$ be a class of representations of X.

(i) $\mathbf{R}^{\mathrm{D}}(X)$ is the subclass of $\mathbf{R}(X)$ containing all dense representations.
(ii) If we fix κ to be an infinite cardinal, then $\mathbf{R}_\kappa(X)$ is the subclass of $\mathbf{R}(X)$ containing all representations where the topological space D has a base with cardinality bounded by κ.

For domain representations, the above definition of representations of bounded size can be rephrased as a bound on the set of compact elements. For example, $\mathbf{DRep}_\kappa(X)$ is the class of all κ-based domain representations, that is, representations where the cardinality of the compact elements of the domain is bounded by κ.

The most interesting class of domain representations with bounded cardinality is, of course, the class of countably based domain representations, that is, $\mathbf{DRep}_\omega(X)$, since these are the ones to which effectivity can be applied.

We will now repeat some of the results from Blanck (2000). The following proposition sums up the results in Blanck (2000, Section 4) and shows why we may restrict our attention to representations from algebraic cpos.

Proposition 2.10. *Let $(D, D^{\mathrm{R}}, \rho_E)$ be a continuous cpo representation of X. Then there is a canonical algebraic cpo representation $(E, E^{\mathrm{R}}, \rho_E)$ retaining the properties of quotient, retract and density if they are present in the original representation.*

For full proofs of the following theorems, see Blanck (2000, Theorems 5.4, 5.6 and 9.3), respectively.

Theorem 2.11. *Any T_0 space has a dense retract domain representation.*

Proof. We will only give the construction briefly here. Let X be a T_0 space, and let B be a base for X that is closed under finite intersections. Let B' be the non-empty sets of B together with the set X. Ordering B' by reverse inclusion gives a cusl, and the ideal completion of this cusl is a domain D. The function $\eta : X \to D$ given by $\eta(x) = \{U \in B' : x \in U\}$ is an embedding.

It just remains to show that $(D, \eta[X], \eta^{-1}, \eta)$ is a dense retract domain representation of X. □

Remark 2.12. By allowing B' in the above construction to contain the empty set, the domain representation will have a compact top element but will otherwise be identical. Constructing the domain representation of X as above, we get a non-dense (since the top element is not in $\eta[X]$) retract domain representation of the space X.

Theorem 2.13. *A space with a retract cpo representation is a T_0 space.*

Theorem 2.14. *Let $(D, D^{\mathrm{R}}, \rho_D)$ be a dense domain representation of X, and let $(E, E^{\mathrm{R}}, \rho_E, \eta_E)$ be a retract domain representation of Y. Then every continuous function $f : X \to Y$ is represented by some continuous function $\bar{f} : D \to E$.*

Proof. The construction of $\bar{f}$ is done in two steps:

1 Let $f' = \eta_E f \rho_D$. The function $f' : D^{\mathrm{R}} \to E$ is a continuous function representing f.

2 The function f' is extended to a function $\bar{f}:D_c \to E$ by $\bar{f}(a) = \sqcap f'[\uparrow a \cap D^{\mathrm{R}}]$. The infimum is well defined since $\uparrow a \cap D^{\mathrm{R}}$ is non-empty by density of D^{R}, and non-empty infima exist in consistently complete cpos. Clearly, $\bar{f}$ is monotone, and hence has a unique extension to D.

It just remains to show that $\bar{f}$ is an extension of f', that is, that $\bar{f}(d) = f'(d)$ for $d \in D^{\mathrm{R}}$. □

Remark 2.15. Density is needed in order to give a well-defined infimum in the second step of the construction of the lifting. The same result can also be achieved if E has a top element, since we may define the infimum of the empty set to be the top element of E. Hence, the above result could be stated for arbitrary D and a retract representation E with a top element.

Other lifting results where density of D is not required can be found in Normann (2001), Blanck (2001) and Køber (2007).

2.3. *Partial domain functions*

We will use the notion of continuous partial domain functions that was introduced in Dahlgren (2007b).

Definition 2.16. Let D and E be domains. A *continuous partial function* from D to E is a pair (S, f) where $S \subseteq D$ is a non-empty closed subset of D and f is a strict continuous function from S to E.

Note that a total domain function is not necessarily a partial domain function. In fact, only strict total functions are partial functions. The strictness of partial functions is required to ensure that composition is always defined. (Allowing totally undefined partial functions would solve composition but means that the category of domains with partial functions does not have a terminal object. The empty domain might be added as the terminal object of the category, but that would make the definition of products hard.)

The closed set S is downwards closed and must therefore contain $\perp$. The closed set of principal interest to us is the closure of the totality of a domain representation, that is, the closed subset S containing the totality of the domain and such that the totality is dense in S.

Recall that a common interpretation of the ordering relation of a domain representation is that it corresponds to information. Higher up in the domain means more information about an object of the space. So a non-dense representation can be viewed as a representation that contains non-consistent information or ‘garbage’. In practice, it is sometimes desirable to cut away this non-consistent information, that is, to restrict to a substructure (see, for example, Blanck (1997b, Lemma 2.28)).

This restriction can be done in general for domain representations by taking the ideal completion over all approximations of representing elements. Formally, for a domain representation $(D, D^{\mathrm{R}}, \rho)$ of X, let

$$D_{\mathrm{c}}^{\mathrm{D}} = \{a \in D_{\mathrm{c}} : \uparrow a \cap D^{\mathrm{R}} \neq \varnothing\}$$

and

$$D^{\mathrm{D}} = \{\bigsqcup A : A \subseteq D^{\mathrm{D}}_{\mathrm{c}} \text{ is directed}\}.$$

Clearly, D^{D} is a closed subset of D. Ordering D^{D} by the order of D makes D^{D} into a domain. The set D^{D} contains D^{R}. Thus, $(D^{\mathrm{D}}, D^{\mathrm{R}}, \rho)$ is a dense domain representation of X, which we will refer to as the *dense part* of D.

Proposition 2.17. Let D be a domain representation in $\mathbf{DRep}(X)$ such that $\perp \notin D^{\mathrm{R}}$, and let E be a retract domain representation of Y. Then any continous function $f : X \to Y$ is representable by a continuous partial domain function $\bar{f} : D \rightharpoonup E$, where $\bar{f}$ is defined on D^{D}.

Proof. By Theorem 2.14, there exists a continuous functions $f' : D^{\mathrm{D}} \to E$. Construct a strict function $\bar{f}$ by

$$\bar{f}(c) = \begin{cases} \perp, & \text{if } c = \perp; \\ f'(c), & \text{otherwise}. \end{cases}$$

As $\perp \notin D^{\mathrm{R}}$, we have that $\bar{f}(x) = f'(x)$ for all $x \in D^{\mathrm{R}}$. □

The restriction that $\perp \notin D^{\mathrm{R}}$ is not severe since any domain representation $(D, D^{\mathrm{R}}, \rho)$ can be lifted to a domain representation $(D_{\perp}, D^{\mathrm{R}}, \rho)$ that satisfies the condition.

3. Reducibility

In order to study representations and their applicability to various tasks, we will now give a notion of reduction between representations of a fixed space. For representations D and E of a space X, we have that D reduces to E if the representation function of D factors through the representation function of E, that is, if there is a function $\phi : D \to E$ such that the following diagram commutes[†].

$$\begin{tikzcd}
D \arrow[rr, "\phi"] & & E \\
D^{\mathrm{R}} \arrow[u, hook, "\iota"] \arrow[rr, "\phi"] \arrow[dr, "\rho_D"'] & & E^{\mathrm{R}} \arrow[u, hook', "\iota"'] \arrow[dl, "\rho_E"] \\
& X &
\end{tikzcd}$$

Another way of interpreting this diagram is that the identity on X is representable, see Lemma 3.3.

We will now give a formal definition of reducibility between representations, which can be compared with Weihrauch (2000, Definition 2.3.2).

Definition 3.1. Let $D = (D, D^{\mathrm{R}}, \rho_D)$ and $E = (E, E^{\mathrm{R}}, \rho_E)$ be representations of a space X.

[†] A weaker notion of reduction is obtained by relaxing the requirement that the continuous map $\phi : D^{\mathrm{R}} \to E^{\mathrm{R}}$ extends to a total function on D. The weak version of effective reductions is considered in Dahlgren (2007a).

(i) A *reduction* of D to E is a function $\phi : D \to E$ such that $\phi[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$ and $\rho_D = \rho_E \phi$.
(ii) For a class Φ of functions, we write $D \leqslant_{\Phi} E$ if there exists a reduction ϕ of D to E in Φ.
(iii) Representations D and E are Φ-*equivalent*, $D \equiv_{\Phi} E$, if $D \leqslant_{\Phi} E$ and $E \leqslant_{\Phi} D$.

The subscript Φ will often be replaced by something more mnemonic, for example, when Φ is the class of all continuous functions, we write $\leqslant_{\mathrm{c}}$. Any class Φ considered contains all identities and is closed under composition. With this assumption on Φ, we have the following result.

Lemma 3.2. Let Φ be a class of functions containing all identities and closed under composition. Then the relation $\leqslant_{\Phi}$ is a pre-order and the relation $\equiv_{\Phi}$ is an equivalence relation.

Proof. The reduction relations are reflexive since the identity is a function reducing a representation to itself. Transitivity follows by composition. □

Lemma 3.3. Let D and E be representations of X. Then $D \leqslant_{\Phi} E$ if and only if the identity function on X is represented by some $\phi : D \to E$ in Φ.

Proof. Any reduction function represents the identity function on X. Any function representing the identity on X is a reduction function. □

The natural objects to study reductions on are the equivalence classes under $\equiv_{\Phi}$. These have a well-defined partial order induced by the pre-order $\leqslant_{\Phi}$. We call this structure the spectrum over a space.

Definition 3.4. A Φ-*spectrum* over a topological space X, written $\mathrm{Spec}(X, \mathscr{D}, \leqslant_{\Phi})$, is the quotient $\mathscr{D}/{\equiv_{\Phi}}$ ordered by $\leqslant_{\Phi}$, where $\mathscr{D}$ is a class of representations of the space X and Φ is a class of reduction functions.

Spectrums are built in the same way as *degree structures* are built in Computability Theory.

We call a largest element of a spectrum *universal* as any question that can be answered in a universal representation can be answered in any representation by reducing the problem to the universal representation.

Definition 3.5. Given a reduction relation $\leqslant_{\Phi}$, we say that a representation D is Φ-*universal* in a class $\mathscr{D}$ if $E \leqslant_{\Phi} D$ for all $E \in \mathscr{D}$.

3.1. *Continuous reductions*

We give some basic results for c-reductions, that is, continuous (total) reductions, of domain representations, that is, let us study $\mathrm{Spec}(X, \mathbf{DRep}(X), \leqslant_{\mathrm{c}})$.

Theorem 3.6. Let D be a dense domain representation of X and E be a retract domain representation of X. Then D continuously reduces to E.

Proof. By Theorem 2.14, the identity function can be lifted to a continuous domain function. By Lemma 3.3, D reduces to E. $\square$

Note that all dense retract representations belong to the same equivalence class.

Corollary 3.7. Dense retract domain representations are unique up to $\equiv_{\mathrm{c}}$.

Proof. The result is immediate. $\square$

The equivalence class of dense retract domain representations is central because any continuous function is representable if the spaces have dense retract domain representations by Theorem 2.14, and because dense retract domain representations exist for all T_0 spaces by Theorem 2.11. However, note that the equivalence class of dense retract representations also contains non-dense retract representations.

The following lemma shows that the retract property is invariant under reductions.

Theorem 3.8. Let $D \leqslant_{\mathrm{c}} E$ be representations of X. If D is a retract representation, then so is E.

Proof. Let $(D, D^{\mathrm{R}}, \rho_D, \eta_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ be the representations. By reducibility, there exists a continuous $\phi : D \to E$ such that $\rho_D = \rho_E\, \phi$.

$$\begin{tikzcd}
D \arrow[rr, "\phi"] & & E \\
D^{\mathrm{R}} \arrow[u, hook, "\iota"] \arrow[rr, "\phi"] \arrow[dr, "\rho_D"] & & E^{\mathrm{R}} \arrow[u, hook', "\iota"'] \arrow[dl, bend left, "\rho_E"] \\
& X \arrow[ul, bend left, "\eta_D"] \arrow[ur, dashed, "\eta_E"] &
\end{tikzcd}$$

Let $\eta_E = \phi\, \eta_D$. Then

$$\rho_E\, \eta_E = \rho_E\, \phi\, \eta_D = \rho_D\, \eta_D = \mathrm{id}_X\,,$$

which shows that $(E, E^{\mathrm{R}}, \rho_E, \eta_E)$ is a retract representation. $\square$

The following examples show that the spectrum $\mathrm{Spec}(X, \mathbf{DRep}(X), \leqslant_{\mathrm{c}})$ is, in general, non-trivial. The first example shows that for non-discrete spaces there exist domain representations that are not retracts and hence are strictly below the equivalence class containing dense retract representations.

Example 3.9. Let X be a non-discrete space. Clearly, the flat domain $X_\perp$ gives a dense domain representation $(X_\perp, X, \mathrm{id})$ of X. The only possible embedding $\imath : X \to X_\perp$ is not continuous since there exists a subset $A \subseteq X$ that is not open in X, whereas A is open in $X_\perp$.

The interested reader can (given a suitable choice principle) show that if the space X is discrete, then any domain representation of X is a retract.

The following example shows that there exist non-dense retract representations that cannot be continuously reduced to any dense retract representation; and hence are strictly above the equivalence class containing dense retract representations.

Example 3.10. Let X be a T_0 space containing at least two points separable by open sets. By Theorem 2.11, there exists a dense retract domain representation $(D, \eta[X], \rho, \eta)$ of X, and by Remark 2.12, there also exists a non-dense retract domain representation $(D_\top, \eta[X], \rho, \eta)$ with a compact top element.

The embedding $\iota : D \to D_\top$ is a continuous reduction of D to $D_\top$.

That X contains two points separable by open sets implies that the domain D contains non-consistent compact elements. Hence, a continuous, and therefore monotone, function cannot reduce $D_\top$ to D. Thus, $D <_{\mathrm{c}} D_\top$.

The above two examples can be summarised by saying that unless the topology of the space X is extremely fine or extremely coarse there exist representations strictly below and strictly above, respectively, the equivalence class of dense retract representations.

The following example shows that the equivalence class of any dense representations also contain non-dense representations.

Example 3.11. Let $(D, D^{\mathrm{R}}, \rho)$ be a dense representation of X. Construct a new domain representation by taking the disjoint sum of D and the trivial domain $\{\bot\}$ consisting of the bottom element only, that is, $(D + \{\bot\}, D^{\mathrm{R}}, \rho)$. Clearly, this representation is non-dense.

The embedding of D into the disjoint sum is a continuous reduction. Conversely, the projection of the disjoint sum onto D (mapping all of the right-hand domain to bottom in D) is a continuous reduction of the disjoint sum to D. Hence $D \equiv_{\mathrm{c}} D + \{\bot\}$.

Continuous equivalence of domain representations implies that the same topological algebras may be represented.

Theorem 3.12. Let $D = (D, D^{\mathrm{R}}, \rho_D)$ and $E = (E, E^{\mathrm{R}}, \rho_E)$ be c-equivalent representations of X. Any operation $f : X^n \to X$ representable in D is also representable in E.

Proof. Let $\phi : D \to E$ and $\psi : E \to D$ be the reductions. Assume that $\bar{f} : D^n \to D$ represents $f : X^n \to X$. Define $\hat{f} : E^n \to E$ by

$$\hat{f}(e_1, \ldots, e_n) = \phi(\bar{f}(\psi(e_1), \ldots, \psi(e_n))),$$

which is continuous as it is the composition of continuous functions. Note that $\hat{f}[E^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$ for all $e_1, \ldots, e_n \in E^{\mathrm{R}}$. We have

$$\begin{aligned}
\rho_E(\hat{f}(e_1, \ldots, e_n)) &= \rho_E(\phi(\bar{f}(\psi(e_1), \ldots, \psi(e_n)))) \\
&= \rho_D(\bar{f}(\psi(e_1), \ldots, \psi(e_n))) \\
&= f(\rho_D(\psi(e_1)), \ldots, \rho_D(\psi(e_n))) \\
&= f(\rho_E(e_1), \ldots, \rho_E(e_n)),
\end{aligned}$$

showing that $\hat{f}$ represents f. □

3.2. *Universal and retract representations*

Since the retract property is closed upwards in a spectrum, it is a notion of being large in the spectrum, but clearly, so also is the notion of universality. We will now compare these two notions.

Theorem 3.13. Consider the spectrum $\mathrm{Spec}(X, \mathscr{D}, \leqslant_{\mathrm{c}})$, where $\mathscr{D}$ is a class of domain representations. If there exists a retract representation in $\mathscr{D}$, then any c-universal representation is a retract representation.

Proof. Let $D \in \mathscr{D}$ be a retract representation and let $E \in \mathscr{D}$ be a c-universal representation. By universality, $D \leqslant_{\mathrm{c}} E$, so, by Theorem 3.8, E is a retract representation. □

There exist universal representations in $\mathbf{DRep}(X)$ when X is a T_0 space. These representations have compact top elements that are not representing. The top element could be interpreted as explicit inconsistent information.

Proposition 3.14. Any T_0 space X has a universal domain representation $(D_{\top}, D^{\mathrm{R}}, \rho)$ in $\mathbf{DRep}(X)$ with respect to $\leqslant_{\mathrm{c}}$.

Proof. Construct the domain representation $(D_{\top}, D^{\mathrm{R}}, \rho)$ as indicated in Remark 2.12. Reductions from any domain representation E of X can now be constructed as indicated in Remark 2.15. □

Restricting the representations to be dense, there are universal representations again, and include the dense retract representations.

Theorem 3.15. Let X be a T_0 space. There exists a c-universal representation of X in $\mathbf{DRep}^{\mathrm{D}}(X)$; and a representation is c-universal if and only if it is a retract.

Proof. By Theorem 2.11, there exists a dense retract domain representation D of the space, and by Theorem 3.6, any dense representation reduces to D, so D is universal, and by Corollary 3.7, any dense retract is universal.

The converse is Theorem 3.13. □

3.3. *Partial reductions*

We now turn our attention to $\mathrm{Spec}(X, \mathbf{DRep}(X), \leqslant_{\mathrm{cp}})$, where $\leqslant_{\mathrm{cp}}$ denotes reductions by continuous partial domain functions. The first result is that any domain representation is cp-equivalent to its dense part.

Proposition 3.16. A domain representation $(D, D^{\mathrm{R}}, \rho_D)$ of X is cp-equivalent to its dense part $(D^{\mathrm{D}}, D^{\mathrm{R}}, \rho_D)$.

Proof. The partial map $\mathrm{id} : D \rightharpoonup D^{\mathrm{D}}$ defined on the closed set D^{D} is a partial domain function reducing D to D^{D}. The embedding $\imath : D^{\mathrm{D}} \to D$ is a partial continuous reduction of D^{D} to D (which happens to be total). Hence $D \equiv_{\mathrm{cp}} D^{\mathrm{D}}$. □

The proposition implies that any element of the spectrum $\mathrm{Spec}(X, \mathbf{DRep}(X), \leqslant_{\mathrm{cp}})$ will contain a dense representation. This simplifies the structure of the spectrum as, in general, there will be fewer equivalence classes.

Lemma 3.17. Let $D = (D, D^{\mathrm{R}}, \rho_D)$ and $E = (E, E^{\mathrm{R}}, \rho_E)$ be domain representations in $\mathbf{DRep}(X)$. If $\perp \notin D^{\mathrm{R}}$, then $D \leqslant_{\mathrm{c}} E$ implies $D \leqslant_{\mathrm{cp}} E$.

Proof. Partial functions are strict so any partial reduction $f : D \rightharpoonup E$ must send $\bot_D$ to $\bot_E$. But if $\bot_D \in D^{\mathrm{R}}$ and $\bot_E \notin E^{\mathrm{R}}$, then $f[D^{\mathrm{R}}] \nsubseteq E^{\mathrm{R}}$, contradicting the fact that f is a reduction.

If the bottom element $\bot_D \notin D^{\mathrm{R}}$, then any total reduction function $f : D \to E$ can be made into a continuous strict domain function (that is, an everywhere defined continuous partial domain function) simply by changing its value at $\bot_D$ if necessary. ☐

To avoid the special case of $\bot$ being a representing element, we let $\mathscr{D} \subseteq \mathbf{DRep}(X)$ be a subclass of domain representations where $\bot$ never is a representing element. This is not a severe restriction as it can be circumvented by lifting all domain representations. Also note that if the space X is T_1 and contains two or more points, then $\bot$ cannot be a representing element.

Theorem 3.18. A domain representation D of X is cp-universal in $\mathrm{Spec}(X, \mathscr{D}, \leqslant_{\mathrm{cp}})$ if and only if D is a retract representation.

Proof. Let D be a retract domain representation and E be an arbitrary domain representation of X. By Proposition 2.17, the identity function on X can be lifted to a continuous partial domain function $\phi : E \rightharpoonup D$ defined on E^{D}. Now, by Lemma 3.3, $E \leqslant_{\mathrm{cp}} D$, showing that D is universal.

For the other direction, assume that D is universal. By Theorem 2.11, there exists a dense retract representation E, and, by universality, $E \leqslant_{\mathrm{cp}} D$. But since E is dense, the reduction is in fact a (total) continuous reduction. Hence, by Lemma 3.8, D is a retract representation. ☐

We note some easy consequences of the above result.

Corollary 3.19. Retract domain representations in $\mathscr{D}$ are unique up to $\equiv_{\mathrm{cp}}$.

Proof. The result is immediate. ☐

Corollary 3.20. For a dense domain representation D of X, the following are equivalent:

(i) D is universal in $\mathrm{Spec}(X, \mathscr{D}^{\mathrm{D}}, \leqslant_{\mathrm{c}})$.
(ii) D is a retract representation.
(iii) D is universal in $\mathrm{Spec}(X, \mathscr{D}, \leqslant_{\mathrm{cp}})$.

Proof. The result follows from Theorems 3.15 and 3.18. ☐

Corollary 3.21. If D is c-universal in $\mathscr{D}$, then D^{D} is c-universal in $\mathscr{D}^{\mathrm{D}}$ and cp-universal in $\mathscr{D}$.

Proof. By Theorem 3.15, D is a retract, so D^{D} is a dense retract. The result then follows by Corollary 3.20. ☐

4. Cantor–Weihrauch domain representations

4.1. *Cantor and Baire domain representations*

Let Σ be an *alphabet*, that is, a finite set of symbols containing at least two symbols. The Kleene star is an operator giving the set of all finite sequences over a set, so Σ^* is the space of all finite sequences over Σ. Let Σ^ω denote the set of all countably infinite sequences over Σ. Concatenation of sequences is denoted by juxtaposition.

Cantor space is the set Σ^ω with a base for the topology given by the sets

$$\{p\sigma : \sigma \in \Sigma^\omega\}, \text{ for } p \in \Sigma^* .$$

Note that the constructed Cantor space does not depend upon the choice of the alphabet Σ, that is, even though the alphabets may differ, the Cantor spaces are homeomorphic.

Baire space is the set $\mathbb{N}^\omega$ with topology defined as for Cantor space.

We will now build domain representations of Cantor and Baire space. The *Cantor domain* over Σ, $\mathscr{C}_\Sigma$, is the set $\Sigma^* \cup \Sigma^\omega$ with the prefix ordering. We will assume some fixed alphabet Σ and drop the subscript Σ for the Cantor domain. It is easy to check that the Cantor domain is an algebraic cpo with Σ^* as its set of compact elements. Two elements of the Cantor domain are consistent only if one is a prefix of the other, so it is clearly consistently complete. There exists a numbering of Σ^* making the Cantor domain into an effective domain. The *Baire domain* $\mathscr{B}$ is constructed likewise.

Cantor and Baire spaces are embedded densely into the Cantor and Baire domains, respectively, as their non-compact elements, so we have the following result, which we state without proof.

Proposition 4.1. The Cantor and Baire domains are effective dense retract domain representations of the Cantor and Baire spaces, respectively.

Definition 4.2. A *Cantor domain representation* of a space X is a domain representation $(\mathscr{C}, \mathscr{C}^{\mathrm{R}}, \rho)$ where the domain is the Cantor domain. Similarly, *Baire domain representation* uses a Baire domain.

Cantor or Baire domain representations exist for many spaces, where, obviously, the cardinality of the space cannot exceed that of the continuum. For a space X, let $\mathbf{CRep}(X)$ and $\mathbf{BRep}(X)$, respectively, denote the classes of Cantor and Baire domain representations of X. Clearly,

$$\mathbf{CRep}(X) \subseteq \mathbf{BRep}(X) \subseteq \mathbf{ERep}(X) \subseteq \mathbf{DRep}_\omega(X) \subseteq \mathbf{DRep}(X),$$

where $\mathbf{ERep}(X)$ is the class of *effective* domain representations of X.

Cantor and Baire domain representations are not retract representations unless the space is zero-dimensional, that is, has a clopen base.

Proposition 4.3. If $(\mathscr{C}, \mathscr{C}^{\mathrm{R}}, \rho, \eta)$ is a retract representation of X, and $\mathscr{C}^R \in \Sigma^\omega$, then X must be zero-dimensional.

Proof. Cantor space is zero-dimensional and so has a clopen basis $\mathscr{B}$. By continuity of η, we have $\{\eta^{-1}[B] : B \in \mathscr{B}\}$ is a clopen basis for X. So X is zero-dimensional. □

4.2. *TTE representations*

In TTE, there are two kinds of *naming systems* for a space X. A *notation* is an onto partial map from the discrete space Σ^* to X, and a *representation* (in TTE terminology) is an onto partial map from Σ^ω to X. Baire space was used in Weihrauch (1987), but the more recent Weihrauch (2000) uses Cantor space, so we will primarily use Cantor space in our discussion – it is easy to check that our setting applies equally well to Baire space representations.

TTE notations can be modelled as domain representations using the flat domain $\Sigma^*_\perp$. TTE representations can be modelled as Cantor domain representations, where $\mathscr{C}^{\mathrm{R}}$ is a subset of Σ^ω. Formally, we have the following lemmas.

Lemma 4.4. TTE notations are in one-to-one correspondence with domain representations where the domain is $\Sigma^*_\perp$ and the totality on the domain does not contain $\perp$.

Proof. A TTE notation of X, that is, a surjective partial map $\nu : \Sigma^* \rightharpoonup X$, gives rise to a domain representation $(\Sigma^*_\perp, \operatorname{dom} \nu, \nu)$ of X.

A domain representation $(\Sigma^*_\perp, {\Sigma^*_\perp}^{\mathrm{R}}, \rho)$ of X where the totality does not contain $\perp$ implies that ρ is a TTE notation since ${\Sigma^*_\perp}^{\mathrm{R}}$ is a subset of Σ^*. □

Lemma 4.5. Any TTE representation gives rise to a Cantor domain representation.

Proof. A TTE representation $\delta : \Sigma^\omega \rightharpoonup X$ of X gives rise to the domain representation $(\mathscr{C}, \mathscr{C}^{\mathrm{R}}, \delta)$, where $\mathscr{C}^{\mathrm{R}} = \operatorname{dom} \delta$. □

In TTE, an arbitrary choice of $\mathscr{C}^{\mathrm{R}}$ is not considered, so the notion of Cantor domain representations is wider than the notion of TTE representations, though no more expressive, as is shown below. The result is originally due to Stoltenberg-Hansen and Tucker (1999b), further results in this direction can be found in Bauer (2002). Note that some spaces are easier to give Cantor domain representations than TTE representations, for example, the Cantor domain itself is trivially represented by itself with the identity, but requires some encoding to make into a TTE representation.

Theorem 4.6. The class of spaces that have TTE representations coincides with the class of spaces that have countably based domain representations.

Proof. Any space that has a TTE representation has a domain representation by Lemma 4.5.

For the other direction, consider a countably based domain representation $(D, D^{\mathrm{R}}, \rho)$ of a space X. We will construct a Cantor domain representation of D, and, in turn, a Cantor domain representation of X. Let $\alpha : \omega \to D_{\mathrm{c}}$ be a numbering of D_{c}, and let $\Sigma = \mathbf{2} = \{0, 1\}$. Define an encoding function $\iota : \mathbb{N} \to \mathbf{2}^*$ by

$$\iota(n) = 01^n 0 .$$

Also, define $\gamma : \mathscr{C} \to \mathscr{P}(\mathbb{N})$ by

$$\gamma(s) = \{n : \imath(n) \text{ is a substring of } s\},$$

that is, γ collects all natural numbers n encoded in the sequence s.

Define the representation function $\delta : \mathscr{C}_c \to D_c$ by

$$\delta(s) = \bigsqcup \alpha(\gamma(p)),$$

where p is the longest prefix of s such that $\alpha(\gamma(p))$ is directed. Clearly, δ is a monotone map, so it has a unique continuous extension to $\mathscr{C}$. Let $\mathscr{C}^R$ be the sequences $s \in \mathbf{2}^\omega$ such that $\gamma(s)$ is directed. Then $(\mathscr{C}, \mathscr{C}^R, \delta)$ is a Cantor domain representation of D.

By composition, we have that $(\mathscr{C}, \delta^{-1}[D^R], \rho\delta)$ is a Cantor domain representation of X. □

TTE also comes with a notion of reduction between naming systems (Weihrauch 2000, Definition 2.3.2), which are continuous partial functions. We will see that these reductions correspond to (total) continuous reductions in our sense.

Definition 4.7. Let $\delta : \Sigma^\omega \rightharpoonup X$ and $\epsilon : \Sigma^\omega \rightharpoonup X$ be TTE representations of X. A function $f : \Sigma^\omega \rightharpoonup \Sigma^\omega$ *reduces* δ to ϵ if $\delta(x) = \epsilon f(x)$ for all $x \in \operatorname{dom}\delta$.

Lemma 4.8. If $f : \Sigma^\omega \rightharpoonup \Sigma^\omega$ is a continuous reduction of δ to ϵ, then $f' : \mathscr{C} \to \mathscr{C}$ reduces $(\mathscr{C}, \operatorname{dom}\delta, \delta)$ to $(\mathscr{C}, \operatorname{dom}\epsilon, \epsilon)$, where f' is defined for compact c by

$$f'(c) = \sqcap f[\uparrow d \cap \operatorname{dom}\delta],$$

where d is the maximal prefix of c such that $\uparrow d \cap \operatorname{dom}\delta \neq \emptyset$.

Proof. Note that the construction of f' is very similar to the second step of the construction used to lift functions to domain representations. As we are restricted to the Cantor domain, we can avoid the need to have $\operatorname{dom}\delta$ dense in $\mathscr{C}$ by carefully defining the function value for non-consistent approximations to be as small as possible while making f' monotone.

The monotone function f' has a unique continuous extension to all of $\mathscr{C}$. We leave the straightforward proof that $f'(x) = f(x)$ for all $x \in \operatorname{dom}\delta$ to the reader. □

We leave the reader to check that reductions between any naming systems (both notations and representations) correspond to continuous (total) reductions in our sense.

The above implies that we can study TTE representations as the spectrum Spec(X, $\mathbf{CRep}(X), \leqslant_c$), which can be identified with a substructure of both Spec($X, \mathbf{DRep}_\omega(X), \leqslant_c$) and Spec($X, \mathbf{DRep}_\omega(X), \leqslant_{cp}$).

4.3. *Cantor–Weihrauch domain representations*

We will now consider a standard construction of Cantor domain representations. For this section we fix the underlying alphabet Σ of Cantor space and the Cantor domain $\mathscr{C}$ to be $\mathbf{2} = \{0, 1\}$.

The following definition is due to Weihrauch, except that we use a numbering of the subbase (from the natural numbers) rather than a notation (from the set of finite words).

Definition 4.9. An *effective topological space* is a triple $S = (X, \sigma, \alpha)$, where X is a non-empty T_0 space, σ is a countable subbase for X and $\alpha : \omega \to \sigma$ is a semicomputable numbering of the subbase.

Let $S = (X, \sigma, \alpha)$ be an effective topological space.

As in the proof of Theorem 4.6, we define $\imath : \mathbb{N} \to \mathbf{2}^*$ and $\gamma : \mathscr{C} \to \mathscr{P}(\mathbb{N})$ by

$$\imath(n) = 01^n0$$

and

$$\gamma(s) = \{n : \imath(n) \text{ is a substring of } s\}.$$

Define a partial function $\delta_S : \mathscr{C} \to X$ by

$$\delta_S(s) = x, \text{ if } \gamma(s) = \{n : x \in \alpha(n)\}.$$

The function δ_S above is well defined since the topology was assumed to be T_0. The objects for which δ_S is defined are sequences encoding all atomic properties of some point *via* the numbering α of the subbase.

The following notion corresponds to the notion of *standard representation* given in Weihrauch (2000).

Definition 4.10. A *Cantor–Weihrauch domain representation (CW domain representation)* is the Cantor domain representation $(\mathscr{C}, \mathscr{C}^{\mathrm{R}}, \delta_S)$, where $\mathscr{C}^{\mathrm{R}} = \operatorname{dom} \delta_S$.

CW domain representations are not dense if the space X contains more than one point.

Proposition 4.11. A CW domain representation is effective.

Proof. The Cantor domain is an effective domain. □

4.4. *Continuous reductions*

We look here at where CW domain representations belong in the spectrum Spec(X, **DRep**(X), $\leqslant_{\mathrm{c}}$) of domain representations under continuous reductions. In fact, CW domain representations are below a naturally constructed dense retract domain representation. In general, by Proposition 4.3, retract representations do not reduce to CW domain representations.

Let $S = (X, \sigma, \alpha)$ be an effective topological space, and let

$$P = \{X\} \cup \left\{ \bigcap A : A \in \mathscr{P}_{\mathrm{f}}(\sigma),\ \bigcap A \neq \varnothing \right\}.$$

Thus, P is a base for the topology τ on X. Ordered by reverse inclusion, P is a neighbourhood system, and by Blanck (2000, Theorem 5.4), the ideal completion over this neighbourhood system is a dense retract representation $(D, \eta[X], \eta^{-1})$ of X. The representation is not effective in general because intersections need not be computable in the effective topological space S.

Proposition 4.12. The representation $(\mathscr{C}, \mathscr{C}^{\mathrm{R}}, \delta_S)$ is reducible to $(D, \eta[X], \eta^{-1})$.

Proof. Let $c \in \mathscr{C}_{\mathrm{c}}$, that is, c is a finite sequence over **2**. Define $\phi : \mathscr{C}_{\mathrm{c}} \to D$ by

$$\phi(c) = \bigcap\{\alpha(n) : \imath(n) \text{ is a substring of } p\},$$

where p is the longest prefix of c for which the intersection is non-empty. The monotone function ϕ can be extended uniquely to a continuous function $\phi : \mathscr{C} \to D$.

If $s \in \mathscr{C}^{\mathrm{R}}$ and $\delta_S(s) = x$, then $\phi(s)$ is the supremum of basic open sets all of which contain x, so $\phi(s) \sqsubseteq I_x = \{A \in P : x \in A\}$. Any basic open set U containing x is the intersection of a finite set of subbasic open sets and any such finite set of subbasic open sets is encoded into s, so $U \sqsubseteq \phi(s)$, showing that $\phi(s) = I_x$. Thus, $\phi[\mathscr{C}^{\mathrm{R}}] \subseteq \eta[X]$ and $\eta^{-1}\phi = \delta_S$. □

5. Admissible representations

5.1. *Notions of admissibility*

In this section we will consider some of the notions of admissibility considered for (TTE and domain) representations. We will often restrict our attention to domain representations that are limited in size in the sense that the set of compact elements have bounded cardinality.

The classical notion of admissibility in TTE is from Weihrauch (2000) and is as follows.

Definition 5.1. Let X be a second countable T_0 space. A domain representation $D \in \mathbf{DRep}_\omega(X)$ is *W-admissible* if D is c-equivalent to a CW domain representation of X.

Weihrauch's original definition was formulated in the less general setting of representations in $\mathbf{CRep}(X)$, but immediately generalises to the superclass $\mathbf{DRep}_\omega(X)$.

The notion of W-admissibility only applies to spaces that have CW domain representations, and these are the second countable T_0 spaces. Schröder (2002) considers an extended notion of admissibility that includes some spaces that are not second countable.

Definition 5.2. A Cantor domain representation $(\mathscr{C}, \mathscr{C}^{\mathrm{R}}, \rho)$ of X is *S-admissible* if any continuous partial function $\epsilon : \Sigma^\omega \rightharpoonup X$ factors through ρ, that is, if there exists a continuous function $\phi : \Sigma^\omega \rightharpoonup \Sigma^\omega$ such that $\epsilon = \rho\phi$ on $\operatorname{dom}\epsilon$.

Note, that ϵ in the definition is not assumed to be onto X.

Hamrin (2005) generalised the above notion of admissibility to domain representations.

Definition 5.3.

(i) A domain representation $D = (D, D^{\mathrm{R}}, \rho)$ is *κ-admissible* if for each κ-based domain E with dense totality E^{R} and for each continuous function $\epsilon : E^{\mathrm{R}} \to X$ there exists a continuous function $\bar{\epsilon} : E \to D$ such that $\epsilon(x) = \rho\bar{\epsilon}(x)$ for all $x \in E^{\mathrm{R}}$.

(ii) A domain representation is *H-admissible* if it is κ-admissible for all κ.

Note that, as with S-admissibility, the function ϵ in the definition above is not assumed to be a representation function, in particular, it need not be onto.

5.2. *Admissibility as universality*

In this section we will show that the notions of admissibility due to Schröder and Hamrin are all notions of universality in the appropriate spectrum.

W-admissibility is not intrinsically a universality notion as it is defined differently, but W-admissible representations are universal in $\mathrm{Spec}(X, \mathbf{CRep}(X), \leqslant_{\mathrm{c}})$ as they are c-equivalent to universal CW domain representations.

The other notions of admissibility are nearly formulated as universality conditions already, but there is a small difference in that universality just requires representations to factor through the universal representation, but admissible representations require that all continuous maps to the space should factor through the representation (these continuous maps could be viewed as *partial representations* of the space). Thus, we have something to prove for each of the admissibility notions considered. The essential step in these proofs is a disjoint sum construction.

Recall that domain representations $(D, D^{\mathrm{R}}, \rho_D)$ of X and $(E, E^{\mathrm{R}}, \rho_E)$ of Y can be made into a domain representation $(D + E, D^{\mathrm{R}} \cup E^{\mathrm{R}}, \rho_{D+E})$ of the disjoint union $X \uplus Y$ by defining

$$\rho_{D+E}(x) = \begin{cases} \rho_D(d), & \text{if } x = \mathrm{in}_0(d) \\ \rho_E(e), & \text{if } x = \mathrm{in}_1(e). \end{cases}$$

Proposition 5.4. A domain representation $(D, D^{\mathrm{R}}, \rho)$ in $\mathbf{DRep}^{\mathrm{D}}_{\kappa}(X)$ is κ-admissible if and only if it is universal in $\mathbf{DRep}^{\mathrm{D}}_{\kappa}(X)$.

Proof.

($\Rightarrow$): All domain representations in $\mathbf{DRep}^{\mathrm{D}}_{\kappa}$ reduce to a κ-admissible representation D by definition of κ-admissibility, so D is universal.

($\Leftarrow$): Let E be a κ-based domain with a dense totality E^{R}. We need to show that any continuous $\epsilon : E^{\mathrm{R}} \to X$ factors through D.
Clearly, $D + E$ is κ-based. The totality $D^{\mathrm{R}} \cup E^{\mathrm{R}}$ is dense. Continuity of ρ_{D+E} follows from the continuity of ρ_D and ϵ on the disjoint spaces D^{R} and E^{R}. Thus, $(D + E, D^{\mathrm{R}} \cup E^{\mathrm{R}}, \rho_{D+E})$ is a dense κ-based domain representation of X. Since D is universal, there exists a continuous $\phi : D + E \to D$ representing the identity on X.
Let $\iota : E \to D+E$ be the continuous embedding of E into $D+E$. Let $\bar{\epsilon} = \phi\iota : E \to D$. By construction, we have $\rho_D\bar{\epsilon} = \rho_D\phi\iota = \epsilon$, showing that ϵ factors through ρ_D. ☐

Proposition 5.5. A domain representation $(D, D^{\mathrm{R}}, \rho)$ in $\mathbf{DRep}^{\mathrm{D}}(X)$ is H-admissible if and only if it is universal in $\mathbf{DRep}^{\mathrm{D}}(X)$.

Proof. The proof is as above, but without cardinality considerations. ☐

Proposition 5.6. A Cantor domain representation $(\mathscr{C}, \mathscr{C}^{\mathrm{R}}, \rho)$ is S-admissible if and only if it is universal in $\mathbf{CRep}(X)$.

Proof. Assume without loss of generality that the underlying alphabet of Cantor spaces contains the symbols 0 and 1. The disjoint sum of two Cantor domain representations

$(\mathscr{C},\mathscr{C}^{\mathrm{R}}_i,\rho_i)$, $i=0,1$ is the Cantor domain representation $(\mathscr{C},\mathscr{C}^{\mathrm{R}},\rho)$ where

$$\mathscr{C}^{\mathrm{R}} = \{0u : u \in \mathscr{C}^{\mathrm{R}}_0\} \cup \{1u : u \in \mathscr{C}^{\mathrm{R}}_1\},$$

and

$$\rho(u) = \begin{cases} \rho_0(v), & \text{if } u = 0v \\ \rho_1(v), & \text{if } u = 1v\,. \end{cases}$$

The rest of the proof is identical to the proofs above. □

It is known that W-admissibility implies S-admissibility within **CRep**, and, trivially, we have that H-admissibility implies κ-admissibility for any κ.

By characterisation results (Schröder 2002, Theorem 13; Hamrin 2005, Theorem 6.8), we have that spaces have S-admissible representations if and only if they have ω-admissible representations. We would like to relate S-admissibility and ω-admissibility directly using reductions, which would be easy if we had a natural class of domain representations that contains both **CRep** and $\mathbf{DRep}^{\mathrm{D}}_{\omega}$. The obvious choice would be $\mathbf{DRep}_{\omega}$, but representations universal in $\mathbf{DRep}^{\mathrm{D}}_{\omega}(X)$ need not be universal in the superclass $\mathbf{DRep}_{\omega}(X)$.

Theorem 5.7. A sequential space X has an S-admissible representation if and only if it has a c-universal representation in $\mathbf{DRep}_{\omega}$.

Proof. By Bauer (2002, Theorem 5.2), $\mathbf{PQ}_0$ and **AdmSeq** are the same category, also known as the category of qcb-spaces. Hence, if X has an S-admissible representation, there exists an ω-projecting quotient $q : Y \to X$, where Y is a countably based T_0 space. By Proposition 3.14, there exists a c-universal domain representation $(D_{\top}, D^{\mathrm{R}}, \rho_D)$ of Y, which will be countably based since Y is countably based. We show that $(D_{\top}, D^{\mathrm{R}}, q \circ \rho_D)$ is a c-universal representation of X. Let $(E, E^{\mathrm{R}}, \rho_E)$ be a domain representation of X. By the property of ω-projecting, there exists a continuous $g : E^{\mathrm{R}} \to Y$ such that $q \circ g = \rho_E$. Now, (E, E^{R}, g) is a domain representation of Y, and by c-universality, there exists a continuous domain function $\phi : E \to D$ representing the identity on Y, and hence also the identity on X. So $(E, E^{\mathrm{R}}, \rho_E) \leqslant_{\mathrm{c}} (D_{\top}, D^{\mathrm{R}}, q \circ \rho_D)$.

For the other direction, we have by Corollary 3.21 that if $(D, D^{\mathrm{R}}, \rho)$ is a c-universal countably based domain representation of X, its dense part D^{D} (or the dense part of the lifting $D_{\perp}$ if $\perp \in D^{\mathrm{R}}$) is c-universal in $\mathbf{DRep}^{\mathrm{D}}_{\omega}(X)$. By Proposition 5.5, X is ω-admissible, and, by characterisation results, S-admissible. □

6. Representations of the reals

In this section we will look at some different representations of real numbers. The first is the customary interval domain; the second is a substructure of the interval domain that allows for more efficient computations. We will also consider the CW domain representation and other Cantor domain representations of the reals.

Let $\mathscr{R}$ be the ideal completion of all closed rational intervals together with the real line ordered by reverse inclusion. The representing elements $\mathscr{R}^{\mathrm{R}}$ of this domain are all ideals that have singleton intersections; a representing ideal is mapped by $\rho_{\mathscr{R}}$ to the single element of its intersection. Define $\eta_{\mathscr{R}}$ by

$$\eta_{\mathscr{R}}(x) = \{[a,b] : a < x < b,\ a,b \in \mathbb{Q}\}.$$

Lemma 6.1. $(\mathscr{R}, \mathscr{R}^{\mathrm{R}}, \rho_{\mathscr{R}}, \eta_{\mathscr{R}})$ is an admissible representation of the reals.

Proof. A standard proof shows that the representation is a dense retract domain representation. □

Centred dyadic approximations are considered in Blanck (2006) for efficient implementations of exact real arithmetic. These form an interesting substructure of the interval domain.

Definition 6.2. A *centred dyadic interval* is represented by a triple (m, e, s) of the form

$$a = (m \pm e)2^{-s},$$

where the *significant* m and the *exponent* s are integers, and the *error term* e is a natural number. A real x is *approximated* by a if

$$|x - m2^{-s}| \leqslant e2^{-s},$$

or, equivalently,

$$x \in [(m-e)2^{-s}, (m+e)2^{-s}].$$

Fix $j > 0$. A *centred dyadic j-approximation* is a centred dyadic interval where the error term is strictly bounded by 2^j.

Let $\mathscr{R}_{j\text{-cda}}$ be the ideal completion of all centred dyadic j-approximations together with the real line ordered by reverse inclusion. Representing elements $\mathscr{R}^{\mathrm{R}}_{j\text{-cda}}$ are again ideals with singleton intersection and the representing function $\rho_{\mathscr{R}_{j\text{-cda}}}$ is defined as before.

Blanck (2006, Lemma 3.7) shows that $\mathscr{R}_{j\text{-cda}}$ is not a domain for $j > 1$, but that it is a bifinite domain (or SFP-domain). Nevertheless, we will show that with respect to reducibility, $\mathscr{R}_{j\text{-cda}}$ is equivalent to the interval domain. The following lemma shows that even though finite suprema do not exist in general in $\mathscr{R}_{j\text{-cda}}$, there is a sufficiently rich substructure of $\mathscr{R}_{j\text{-cda}}$ where finite suprema do exist.

Lemma 6.3. Within the substructure of all centred dyadic 1-approximations finite suprema exist.

Proof. It is sufficient to show that the supremum of $a = (m \pm 1)2^{-s}$ and $b = (n \pm 1)2^{-t}$ is a centred dyadic 1-approximation. We can assume without loss of generality that $t \geqslant s$.

As the boundaries of a are multiples of 2^{-t} and the radius of b is 2^{-t}, there are only three cases to consider for consistent a and b. If b is a subset of a, the supremum is b. If a and b are touching, that is, have a single point intersection, the supremum is either $(n - 1 \pm 0)2^{-t}$ or $(n + 1 \pm 0)2^{-t}$. In the remaining case, the centre of b is on the boundary of a. Assume that the centre of b is the upper end-point of a, that is, $n2^{-t} = (m+1)2^{-s}$. The supremum of a and b is $((2n-1) \pm 1)2^{-t-1}$. □

We show that the representations based on centred dyadic approximations are not only continuously equivalent to the interval domain, but effectively equivalent.

Theorem 6.4. The representations $\mathscr{R}$ and $\mathscr{R}_{j\text{-cda}}$ are e-equivalent.

Proof. The inclusion map from $\mathscr{R}_{j\text{-cda}}$ to $\mathscr{R}$ represents the identity on the real line, so $\mathscr{R}_{j\text{-cda}}$ reduces to $\mathscr{R}$.

For the other direction, define ϕ on compacts by

$$\phi([a,b]) = \bigsqcup_{1}\{(m \pm 1)2^{-s} : [a,b] \subseteq (m \pm 1)2^{-s},\ 2^{-s} \leqslant 2(b-a)\}$$

if $a \neq b$, and by

$$\phi([a,a]) = \bigsqcup\{(m \pm 1)2^{-s} : a \text{ is approximated by } (m \pm 1)2^{-s}\}$$

otherwise. The supremum in the latter equation is over a directed set. The $\bigsqcup_1$ in the former equation gives finite suprema in the substructure of centred dyadic 1-approximations. We need to show that the supremum is taken over a finite non-empty set. For any interval $[a,b]$ there exists an s such that $b - a < 2^{-s} \leqslant 2(b-a)$, hence there must exist a 1-approximation of the form $(m \pm 1)2^{-s}$ belonging to the set. With a bounded radius, there can only be finitely many 1-approximations containing the interval.

Extend ϕ to a continuous function. Then ϕ represents the identity on the real line. Thus $\mathscr{R}$ reduces to $\mathscr{R}_{j\text{-cda}}$. □

Corollary 6.5. Any algebra $(\mathbb{R}, \{c_i\}_{i \in I}, \{f_k\}_{k \in K})$, where $f_k : \mathbb{R}^{n_k} \to \mathbb{R}$, represented by $\mathscr{R}$ is represented by $\mathscr{R}_{j\text{-cda}}$ and *vice versa.*

Proof. The result follows from Theorems 6.4 and 3.12. □

We denote the Cantor–Weihrauch domain representation of the reals by $C_{\mathbb{R}}$, the Cantor domain representation based on base b expansion of the reals by C_b, and the Cantor domain representation based on signed binary expansion by C_{s2}. Weihrauch (2000) showed that

$$C_b <_{\mathrm{c}} C_{s2} \equiv_{\mathrm{e}} C_{\mathbb{R}}\,.$$

Since all of the above representations are Cantor domain representations and the reals are not zero-dimensional, none of them are retract representations. Thus, by Theorem 3.8, the retract domain representation $\mathscr{R}$ does not continuously reduce to any of these Cantor domain representations.

Going in the other direction, we note that C_{s2} is a dense domain representation, so, by Theorem 3.6, $C_{s2} \leqslant_{\mathrm{c}} \mathscr{R}$. It is an easy exercise to show that this reduction is effective. This gives us the following relations:

$$C_b <_{\mathrm{c}} C_{s2} \equiv_{\mathrm{e}} C_{\mathbb{R}} <_{\mathrm{c}} \mathscr{R}\,.$$

The representation C_b is neither an admissible domain representation nor S-admissible. It is well known that representing the reals by binary expansions is not an appropriate choice when considering computability of operations. For example, neither addition nor multiplication is computable on binary expansions of real numbers. On the other hand, as the binary expansion representation is not a quotient representation, there are (non-continuous) operations that can be represented, for example, finding the 17th bit in the binary expansion. For some applications it might be valuable to remember that although representability of an operation is preserved by equivalences, it is not preserved by reductions.

Signed binary expansions can be used to implement exact real arithmetic, that is, a suitable algebra over the reals. The non-equivalence of C_{s2} and $\mathscr{R}$ seems to hint that the operations that are computable may differ. This is not the case since there exists a domain version D_{s2} of C_{s2} that is e-equivalent to $\mathscr{R}$. Let $\Sigma = \{\overline{1}, 0, 1\}$, and let S be the set of finite signed binary expansions of reals, that is, words over $\Sigma \cup \{.\}$ with exactly one occurrence of '.'. Define an operator $I : S \to \mathscr{P}\mathbb{R}$ that maps a word in $s \in S$ to the closed interval encoded by s, that is, $I(s) = \{s\sigma : \sigma \in \Sigma^{\omega}\}$, where a signed binary expansion $s\sigma$ is interpreted as a real number. Note that it is easy to find the dyadic endpoints of $I(s)$ effectively. Define a preorder $\sqsubseteq$ on S by reverse inclusion of encoded intervals. The ideal completion of this preorder is the domain D_{s2}.

Theorem 6.6. The representations $\mathscr{R}$ and D_{s2} are e-equivalent.

Proof. The map I is an effective map reducing D_{s2} to $\mathscr{R}$. For the other direction, define a map $f : \mathscr{R}_c \to D_{s2}$ by

$$f([a,b]) = \bigsqcup \{s : I(s) \subseteq [a,b]\} .$$

The extension of f to $\mathscr{R}$ is an effective map reducing $\mathscr{R}$ to D_{s2}. □

The difference between D_{s2} and C_{s2} is that signed binary sequences σ and τ are identified in D_{s2} if τ can be obtained from σ by repeated applications of the substitutions $01 = 1\overline{1}$ and $0\overline{1} = \overline{1}1$.

We conclude with a very easy observation.

Theorem 6.7. The cardinality of $\mathrm{Spec}(\mathbb{R}, \mathbf{DRep}(\mathbb{R}), \leqslant_c)$ is infinite.

Proof. Let p be a prime such that p does not divide b. Then the fraction $\frac{1}{p}$ has an infinite expansion in base b. There is no continuous reduction of C_b to C_p since any finite prefix of the expansion of $\frac{1}{p}$ in C_b is not enough to determine if it should be sent to $.0\ldots$ or $.1\ldots$ in C_p. The result follows as there are infinitely many prime numbers. □

As this proof shows, there are infinitely many non-equivalent countably based domain representations of the reals.

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References

Abramsky, S. and Jung, A. (1994) Domain theory. In: Abramsky, S. *et al.* (eds.) *Handbook of Logic in Computer Science*, volume III, Oxford University Press 1–168.

Bauer, A. (2002) A relationship between equilogical spaces and type two effectivity. *Mathematical Logic Quarterly* **48** (supplement 1) 1–15.

Bauer, A., Birkedal, L. and Scott, D. S. (2004) Equilogical spaces. *Theoretical Computer Science* **315** 35–59.

Berger, U. (1996) Density theorems for the domains-with-totality semantics of dependent types. In: Adamek, J., Koslowski, J., Pollara, V. and Struckmann, W. (eds.) *Proceedings of the Workshop Domains II*, Technische Universität Braunschweig.

Berger, U. (1999) Density theorems for the domains-with-totality semantics of dependent types. *Applied Categorical Structures* **7** 3–30.

Birkedal, L., Carboni, A., Rosolini, G. and Scott, D. S. (1998) Type theory via exact categories. In: *Proceedings of the 13th Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press 188–198.

Blanck, J. (1995) *Domain representability of metric spaces*, Licentiate thesis, Uppsala University. U.U.D.M. Report 1995:20.

Blanck, J. (1997a) *Computability on topological spaces by effective domain representations*, Ph.D. thesis, Uppsala University. Uppsala Dissertations in Mathematics **7**.

Blanck, J. (1997b) Domain representability of metric spaces. *Annals of Pure and Applied Logic* **83** 225–247.

Blanck, J. (1999) Effective domain representations of $\mathscr{H}(X)$, the space of compact subsets. *Theoretical Computer Science* **219** 19–48.

Blanck, J. (2000) Domain representations of topological spaces. *Theoretical Computer Science* **247** 229–255.

Blanck, J. (2001) Effectivity of regular spaces. In: Blanck, J., Brattka, V. and Hertling, P. (eds.) Computability and Complexity in Analysis. *Springer-Verlag Lecture Notes in Computer Science* **2064** 1–15.

Blanck, J. (2006) Exact real arithmetic using centred intervals and bounded error terms. *Journal of Logic and Algebraic Programming* **66** 50–67.

Dahlgren, F. (2007a) *Effective Distribution Theory*, Ph.D. thesis, Uppsala University.

Dahlgren, F. (2007b) Partial continuous functions and admissible domain representations. *Journal of Logic and Computation* (to appear).

di Gianantonio, P. (1996) Real number computability and domain theory. *Information and Computation* **127** 11–25.

Edalat, A. (1995a) Domain theory and integration. *Theoretical Computer Science* **151** 163–193.

Edalat, A. (1995b) Dynamical systems, measures, and fractals via domain theory. *Information and Computation* **120** 32–48.

Edalat, A. (1997) Domains for computation in mathematics, physics and exact real arithmetic. *Bulletin of Symbolic Logic* **3** (4) 401–452.

Edalat, A. and Heckmann, R. (1998) A computational model for metric spaces. *Theoretical Computer Science* **193** 53–73.

Edalat, A. and Lieutier, A. (1999) Foundation of a computable solid modelling. In: *Proceedings of ACM's Symposium on Solid Modeling '99*, ACM.

Edalat, A. and Lieutier, A. (2002) Domain theory and differential calculus. In: *LICS'02.*

Ershov, Y. (1973) Theorie der Numerierungen. *Zeitschrift für Math. Log.* **19** (4) 289–388.

Ershov, Y. (1975) Theorie der Numerierungen II. *Zeitschrift für Math. Log.* **21** (6) 473–584.

Ershov, Y. (1977a) Theorie der Numerierungen III. *Zeitschrift für Math. Log.* **23** (4) 289–371.

Ershov, Y. L. (1977b) Model C of partial continuous functionals. In: Gandy, R. O. and Hyland, J. M. E. (eds.) Logic Colloquium 76. *Studies in Logic and Foundations in Mathematics* **87** 455–467.

Hamrin, G. (2005) *Effective Domains and Admissible Domain Representations*, Ph.D. thesis, Department of Mathematics, Uppsala University.

Køber, P. (2007) Uniform domain representations of l^p-spaces. *Mathematical Logic Quarterly* **53** (2) 180–205.

Lietz, P. (2004) *From Constructive Mathematics to Computable Analysis via the Realizability Interpretation*, Ph.D. thesis, TU Darmstadt.

Mal'cev, A. I. (1961) Constructive algebras. I. *Uspehi Mat. Nauk* **16** (3 (99)) 3–60.

Menni, M. and Simpson, A. (2002) Topological and limit-space subcategories of countably-based equilogical spaces. *Mathematical Structures in Computer Science* **12** 739–770.

Normann, D. (1996) A hierarchy of domains with totality, but without density. In: Cooper, S. B., Slaman, T. A. and Wainer, S. S. (eds.) Computability, Enumerability, Unsolvability. *London Mathematical Society Lecture Notes Series* **224** 233–257.

Normann, D. (2001) The continuous functionals of finite type over the reals. In: Keimel, K., Liu, Y. M. and Zhang, G.-Q. (eds.) *Domains and Processes*, Kluwer Academic Publishers 103–124.

Rosolini, G. (2000) Equilogical spaces and filter spaces. *Rendiconti del Circolo Matematico di Palermo (Serie II)* **64**.

Schröder, M. (2002) Extended admissibility. *Theoretical Computer Science* **284** (2) 519–538.

Scott, D. S. (1996) A new category? Domains, spaces and equivalence relations. (Manuscript.)

Stoltenberg-Hansen, V., Lindström, I. and Griffor, E. R. (1994) *Mathematical Theory of Domains*, Cambridge University Press.

Stoltenberg-Hansen, V. and Tucker, J. V. (1985) Complete local rings as domains. CTCS Report 1.85, University of Leeds.

Stoltenberg-Hansen, V. and Tucker, J. V. (1988) Complete local rings as domains. *Journal of Symbolic Logic* **53** 603–624.

Stoltenberg-Hansen, V. and Tucker, J. V. (1991) Algebraic and fixed point equations over inverse limits of algebras. *Theoretical Computer Science* **87** 1–24.

Stoltenberg-Hansen, V. and Tucker, J. V. (1993) Infinite systems of equations over inverse limits and infinite synchronous concurrent algorithms. In: de Bakker, J. W., de Roever, W.-P. and Rozenberg, G. (eds.) Semantics: Foundations and Applications. *Springer-Verlag Lecture Notes in Computer Science* **666** 531–562.

Stoltenberg-Hansen, V. and Tucker, J. V. (1995) Effective algebra. In: Abramsky, S. *et al.* (eds.) *Handbook of Logic in Computer Science*, volume IV, Oxford University Press 357–526.

Stoltenberg-Hansen, V. and Tucker, J. V. (1999a) Computable rings and fields. In: Handbook of computability theory. *Stud. Logic Found. Math.* **140**, North-Holland 363–447.

Stoltenberg-Hansen, V. and Tucker, J. V. (1999b) Concrete models of computation for topological algebras. *Theoretical Computer Science* **219** 347–378.

Stoltenberg-Hansen, V. and Tucker, J. V. (2007) Computability on topological spaces via domain representations (to appear).

Tucker, J. V. and Zucker, J. I. (2000) Computable functions and semicomputable sets on many-sorted algebras. In: Abramsky, S. *et al.* (eds.) *Handbook of Logic in Computer Science*, volume V, Oxford University Press 317–523.

Tucker, J. V. and Zucker, J. I. (2004) Abstract versus concrete computation on metric partial algebras. *ACM Transactions on Computational Logic* **5** (4) 611–668.

Waagbø, G. A. (1997) *Domains-with-totality semantics for Intuitionistic Type Theory*, Ph.D. thesis, University of Oslo.

Weihrauch, K. (1987) Computability. *EATCS Monographs on Theoretical Computer Science* **9** Springer-Verlag.

Weihrauch, K. (2000) *An Introduction to Computable Analysis*, Springer-Verlag.

Weihrauch, K. and Schreiber, U. (1981) Embedding metric spaces into cpo's. *Theoretical Computer Science* **16** 5–24.