Foundations of category theory

Oliver Kullmann
Computer Science Department
University of Wales Swansea

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Introduction

1. We introduce the basic notions of category theory:
    1. categories
    2. functors
    3. natural transformations.
2. We introduce the main generic examples of categories:
    1. the small categories induced by directed graphs, monoids, and quasi-orders;
    2. the large category of sets (and variations);
    3. the category of categories;
    4. functor categories.
3. We introduce main examples of functors.
4. We discuss various attempts to formulate the notion of a category.

Overview

Set theory

- Set theory is the basis of modern mathematics and of all mathematical modelling.
- It is an “infinitely powerful scripting language” with only one type (namely sets) and one basic (specific) relation (namely element-ship “$\in$”); besides that we have the language of first order logic with equality.
- Set theory provides the basic implementations (while category theory is a more abstract layer).
- We have axioms to build new sets from given sets (by combination or selection), and the axiom of infinity guarantees the existence of an infinite set.

Relations

1. In order to model relations, we need **ordered pairs**; for sets x, y let

$$\boldsymbol{(x, y)} := \{\{x\}, \{x, y\}\}.$$

   We have $(x, y) = (x', y') \Leftrightarrow x = x' \wedge y = y'$.
2. A (binary) **relation** is a set of ordered pairs.
3. Let the **domain** and **range** of a relation R be defined by

$$\begin{aligned} \mathbf{dom}(\boldsymbol{R}) &:= \{x \mid \exists\, y : (x, y) \in R\} \\ \mathbf{rg}(\boldsymbol{R}) &:= \{y \mid \exists\, x : (x, y) \in R\}. \end{aligned}$$

4. Instead of "$(x, y) \in R$" one often writes "xRy".
5. A relation R is
   1. **left-unique** if $\forall\, x, x', y : xRy \wedge x'Ry \Rightarrow x = x'$
   2. **right-unique** if $\forall\, x, y, y' : xRy \wedge xRy' \Rightarrow y = y'$.

Maps

1. A **map** is a right-unique relation.
2. $\boldsymbol{f : A \to B}$ if f a map with $\mathrm{dom}(f) = A$ and $\mathrm{rg}(f) \subseteq B$.

Remarks:

- In the set-theoretical sense a “map” is just its “table” of argument-value relations.
- If we have $f : A \to B$ and $f : A' \to B'$, then we have $A = A'$, but B, B' might be different.
- A map f in this sense is nothing else than a “family” $(f_x)_{x \in \mathrm{dom}(f)}$:
  - The argument here is written as index.
  - The domain is written as outer index in the family specification.
- In the category-theoretical context we will later overload the notion of a map, so that also the “codomain” is specified.

Composition of relations and maps

For relations R, S we define the **composition**

$$\boldsymbol{S \circ R} := \{(x, z) \mid \exists\, y : xRy \wedge ySz\}.$$

We have $\mathrm{dom}(S \circ R) \subseteq \mathrm{dom}(R)$ and $\mathrm{rg}(S \circ R) \subseteq \mathrm{rg}(S)$.

Basic properties are:

- Composition of relations is associative, that is, for arbitrary relations R, R', R'' we have

$$R'' \circ (R' \circ R) = (R'' \circ R') \circ R.$$

- Composition of maps yields again a map, that is, for maps f, g also $g \circ f$ is a map.
- And if $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$.

Tuples

We now define (ordered) n-tuples t for $n \in \mathbb{N}$ by induction:

1. Every set ("object") x is a 1-tuple $t = (x) = x$.
2. For $n > 1$ an n-tuple is a pair

$$t = (t', x),$$

   where t' is an $(n-1)$-tuple, while x is a set ("object").

Thus the 2-tuples are exactly the (ordered) pairs. Without going into further details:

- Given an tuple t of length n, for $i \in \{1, \ldots, n\}$ we can access the i-th component of t via t_i.
- Given n objects $x_1, \ldots, x_n$, we can create the n-tuple $t = (x_1, \ldots, x_n)$ with $t_i = x_i$ for $i \in \{1, \ldots, n\}$.

Cartesian products

For sets $A_1, \dots, A_n$, $n \in \mathbb{N}$:

$$\boldsymbol{A_1 \times \cdots \times A_n} := \{(a_1, \dots, a_n) \mid \forall\, i \in \{1, \dots, n\} : a_i \in A_i\}.$$

For an arbitrary family $(A_i)_{i \in I}$ of sets we define a **choice function** as a map f with $\mathrm{dom}(f) = I$ such that $\forall\, i \in I : f(i) \in A_i$.

Now

$$\boldsymbol{\prod_{i \in I} A_i} := \{f \mid f \text{ choice function for } (A_i)_{i \in I}\}.$$

Note that the empty product ($I = \emptyset$) is $\{\emptyset\}$.

Sequences

Tuples other than pairs are of restricted use:

1. there is no empty tuple;
2. given an arbitrary tuple, we cannot recover its length;
3. we cannot mix tuples of different length.

To remove these restrictions, we define "(standardised) sequences":

- An **initial segment** of the set $\mathbb{N}$ of natural numbers is a subset $I \subseteq \mathbb{N}$ such that with $n \in I$ for every $m \in \mathbb{N}$ with $m \le n$ we also have $m \in I$.
- A **(standardised) sequence** is a family with index set an initial segment of $\mathbb{N}$.
- A sequence is **finite** if the index set is finite, and then its **length** is the number of elements of the index set (equivalently for non-empty index sets, the largest element of the index set).

Concatenation of finite sequences

Given two finite sequences a, b of length $m, n \in \mathbb{N}_0$ respectively, the **concatenation**

$$\boldsymbol{a \sqcup b}$$

is a finite sequences of length $m + n$ given by

$$(a \sqcup b)_i := \begin{cases} a_i & \text{if } i \leq m \\ b_{i-m} & \text{if } i > m \end{cases}$$

for $i \in \{1, \ldots, m + n\}$. Main properties ($a, b, c$ are finite sequences):

1. The empty sequence is the neutral element: $\emptyset \sqcup a = a \sqcup \emptyset = a$.
2. Concatenation is associative: $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$.

For further studies

In my opinion, every mathematician (this includes every theoretical computer scientist) should know the basic axioms, definitions and theorems of set theory.

As a thorough introduction into set theory as a foundation of mathematics I recommend:

Nicolas Bourbaki, Elements of Mathematics
Theory of Sets
Springer, 2004
ISBN 3-540-22525-0
softcover reprint of the English translation from 1968.

Directed general graphs

A **directed general graph** (dgg) is a quadruple $G = (V, E, \mathrm{src}, \mathrm{trg})$ such that

1. V is a set (the set of "vertices")
2. E is a set (the set of "directed edges")
3. $\mathrm{src}, \mathrm{trg} : E \to V$ ("source" and "target" of edges).

Remarks:

- In other words, a dgg is a 2-sorted algebraic structure with sorts V, E and with two unary functions of type $E \to V$.
- $V(G) := V$, $E(G) := E$, $\mathrm{src}(G) := \mathrm{src}$, $\mathrm{trg}(G) := \mathrm{trg}$.
- The simplest dgg is the null-dgg $(\emptyset, \emptyset, \emptyset, \emptyset)$ (the unique dgg G with $V(G) = \emptyset$).

Directed general graphs according to the above definition are sometimes called "precategories".

Remarks on the notion of directed graphs

Recall:

1. A **directed graph** is a pair (V, R), where R is an irreflexive relation on V (i.e., $R \subseteq V^2$ with $\mathrm{id}_V \cap R = \emptyset$).
2. A **directed graph allowing loops** is a pair (V, R), where R is a relation on V (i.e., $R \subseteq V^2$).
3. A **general directed graph** is a triple (V, E, γ), where V, E are sets and $\gamma : E \to V^2$.

Now for the notion of a "dgg" as defined above, the map γ is just split into its two components.

(Note that every map $f : X \to Y \times Z$ corresponds 1-1 to pairs of maps (f_1, f_2) with $f_1 : X \to Y$ and $f_2 : X \to Z$ (this is the "universal property" of products(!)).)

Notions from graph theory: Walks

- A (finite) **walk** in a dgg G is a finite sequence

$$W = v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_m$$

for some $m \in \mathbb{N}_0$, vertices $v_0, \ldots, v_m \in V(G)$ and edges $e_1, \ldots, e_m \in E(G)$ such that

$$\forall\, i \in \{1, \ldots, m\} : \operatorname{src}(e_i) = v_{i-1} \wedge \operatorname{trg}(e_i) = v_i.$$

m is the **length** of W, $\mathbf{src}(W) := v_0$, $\mathbf{trg}(W) := v_m$.

- **Concatenation** of walks W, W' is defined if $\operatorname{trg}(W) = \operatorname{src}(W')$, and is obtained from the concatenation of the sequences by removing the (repeated) start vertex from the second walk.
- “Trails” and “paths” are more specialised notions:
  - For trails edges are not repeated,
  - while for paths edges and vertices are not repeated (except possibly of source and target of the walk).

Stripped walks

- A **stripped walk** W' is obtained from a walk W by removing the vertices if the length is not zero and converting it to a tuple.

  That is, from the sequence $W = v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_m$ for $m \geq 1$ we obtain $W' = (e_1, \ldots, e_m)$, while for $m = 0$ we obtain the 1-tuple (v_0).
- For $k \in \mathbb{N}_0$ let $\boldsymbol{W'_k(G)}$ be the set of stripped walks of G of length k:
  1. $W'_0(G) = V(G)$
  2. $W'_1(G) = E(G)$
  3. $W'_2(G) = \{(e, e') \in E(G)^2 : \operatorname{trg}(e) = \operatorname{src}(e')\}$.

Subgraphs

Consider dgg's G and G':

1. G' is a **sub-dgg** of G if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$ and $\mathrm{src}(G') = \mathrm{src}(G) \mid V(G')$ as well as $\mathrm{trg}(G') = \mathrm{trg}(G) \mid V(G')$.
2. G' is a **partial dgg** of G if G' is a sub-dgg of G with $V(G') = V(G)$.
3. G' is an **induced sub-dgg** of G if G' is a sub-dgg of G with $\forall\, e \in E(G) : \mathrm{src}(G)(e), \mathrm{trg}(G)(e) \in V(G') \Rightarrow e \in E(G')$.

Thus induced sub-dgg's are obtained by (only) removing vertices, partial dgg's are obtained by (only) removing edges, and sub-dgg's are obtained by a combination of both processes.

Definition of a category

A category $\mathfrak{C}$ is a triple $\mathfrak{C} = (G, \circ, \mathrm{id})$ such that

- G is a directed general graph
- $\circ : W_2'(G) \to W_1'(G)$
- $\mathrm{id} : V(G) \to E(G)$

fulfilling the following properties (using functional order for the infix-composition):

1. For all $f \in W_2'(G)$ we have $\mathrm{src}(\circ(f)) = \mathrm{src}(f)$ and $\mathrm{trg}(\circ(f)) = \mathrm{trg}(f)$.
2. For all $f \in W_3'(G)$ we have $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$.
3. For all $X \in V(G)$ we have $\mathrm{src}(\mathrm{id}_X) = \mathrm{trg}(\mathrm{id}_X) = X$.
4. For all $X \in V(G)$ and all $f \in E(G)$ we have
   - $\mathrm{src}(f) = X \Rightarrow f \circ \mathrm{id}_X = f$,
   - $\mathrm{trg}(f) = X \Rightarrow \mathrm{id}_X \circ f = f$.

Further notations and our first category

For a category $\mathfrak{C} = (G, \circ, \mathrm{id})$:

- **dgg($\mathfrak{C}$)** $:= G$
- $\circ_{\mathfrak{C}} := \circ$
- **id($\mathfrak{C}$)** $:= \mathrm{id}$

and furthermore

- **Obj($\mathfrak{C}$)** $:= V(G)$ (the **set of objects**)
- **Mor($\mathfrak{C}$)** $:= E(G)$ (the **set of morphisms**)
- **dom**$_{\mathfrak{C}} := \mathrm{src}(G)$ (the **domain**)
- **cod**$_{\mathfrak{C}} := \mathrm{trg}(G)$ (the **codomain**).

The simplest category is the empty category $((\emptyset, \emptyset, \emptyset, \emptyset), \emptyset, \emptyset)$, the unique category with empty object set.

Subcategories

Consider categories $\mathfrak{C}$, $\mathfrak{C}'$:

- $\mathfrak{C}'$ is a **subcategory** of $\mathfrak{C}$ if $\mathrm{dgg}(\mathfrak{C}')$ is a sub-dgg of $\mathrm{dgg}(\mathfrak{C})$, and the maps for composition and identities are obtained by restriction.
- $\mathfrak{C}'$ is a **partial category** of $\mathfrak{C}$ if $\mathfrak{C}'$ is a subcategory of $\mathfrak{C}$ and $\mathrm{dgg}(\mathfrak{C}')$ is a partial dgg of $\mathrm{dgg}(\mathfrak{C})$.
- $\mathfrak{C}'$ is an **induced subcategory** (or a **full subcategory**) of $\mathfrak{C}$ if $\mathfrak{C}'$ is a subcategory of $\mathfrak{C}$ and $\mathrm{dgg}(\mathfrak{C}')$ is an induced sub-dgg of $\mathrm{dgg}(\mathfrak{C})$.

So partial categories have lost only morphisms, while full subcategories have lost only objects. (Subcategories in general may have lost both.)

The free category generated by a dgg

Consider a dgg G. The **free category cat(G)** is the following category:

1. the object set is $V(G)$;
2. the morphism set is the set of all walks in G;
3. domains and codomains are the sources and targets of walks;
4. composition of morphisms is concatenation of walks;
5. the identity of an object X is the walk of length 0 starting at X.

The notion of a monoid

- A **groupoid** is a pair $(V, \circ)$, where V is a set and $\circ : V^2 \to V$ an (internal) binary composition.
- A **semigroup** is a groupoid $(V, \circ)$ such that composition is associative, that is

$$\forall x, y, z \in V : x \circ (y \circ z) = (x \circ y) \circ z.$$

- An element $e \in V$ of a groupoid $(V, \circ)$ is **neutral** if

$$\forall x \in V : e \circ x = x \circ e = x.$$

- A **monoid** is a triple $(M, \circ, e)$ such that $(M, \circ)$ is a semigroup with neutral element e.

The word monoid

For an arbitrary set A let $\boldsymbol{A^*}$ denote the set of finite sequences in A:

- A^* together with concatenation of sequences as composition and the empty sequence as neutral element is a monoid, called
  - the **word monoid** over the alphabet A, or
  - the **free monoid** generated by A.
- Consider the dgg G_A with the single vertex 0 and the edge-set A (these directed edges are then all loops):

  The free category $\text{cat}(G_A)$ is basically the same as the free monoid A^*.

What does "basically the same" mean here?

The endomorphism monoid

Consider a category $\mathfrak{C}$.

1. An **endomorphism** in $\mathfrak{C}$ is a morphism f in $\mathfrak{C}$ with $\mathrm{dom}(f) = \mathrm{cod}(f)$.
2. Obviously the composition of two endomorphisms is again an endomorphism, and the identity morphisms are endomorphisms.
3. Consider an object X in $\mathfrak{C}$. The set

$$\mathbf{End}_{\mathfrak{C}}(\boldsymbol{X}) := \{f \in \mathrm{Mor}(\mathfrak{C}) : \mathrm{dom}(f) = \mathrm{cod}(f) = X\}$$

   of endomorphisms of X is naturally equipped with a monoid structure:
   1. The composition of $\mathrm{End}(X)$ is $\circ_{\mathfrak{C}} \mid \mathrm{End}(X)^2$.
   2. The neutral element of $\mathrm{End}(X)$ is id_X.

(Remark: $\mathrm{End}(X)$ has the **automorphism group** $\mathrm{Aut}(X)$ has sub-group. More in Talk 2.)

First answer

We said that $\mathrm{dgg}(G_A)$ is "essentially the same" as A^*:

Now we see, that actually the monoid A^* is identical to the endomorphism monoid in $\mathrm{dgg}(G_A)$ of the one (and only) object.

Since there is nothing else in $\mathrm{dgg}(G_A)$, "essentially the same" seems to be justified.

Now aren't in general one-object categories and monoids essentially the same?!

Categories with one object

- Monoid $\underline{M} = (M, \circ, e)$ yields a category $\text{cat}(\underline{M})$:
  - the object set is the (singleton) set $\{\emptyset\}$;
  - the morphism set is M;
  - since there is only one object, domains and codomains are always this object;
  - the composition is the composition of the monoid;
  - the identity of the single object is e.
- Given a category $\mathfrak{C}$ with $\text{Obj}(\mathfrak{C}) = \{X\}$, we obtain a monoid $(M, \circ, e)$ via
  - $M := \text{Mor}(\mathfrak{C})$;
  - $\circ := \circ_{\mathfrak{C}}$;
  - $e := \text{id}(\mathfrak{C})(X)$.

The two constructions are "nearly" inverse to each other (only by going from the one-object category to the monoid and back we lose the (single) object). Thus one-object categories and monoids are "essentially" the same.

The notion of a quasi-order

A relation R is called

- **transitive** if $\forall\, x, y, z : xRy \wedge yRz \Rightarrow xRz$
- **symmetric** if $\forall\, x, y : xRy \Rightarrow yRx$
- **antisymmetric** if $\forall\, x, y : xRy \wedge yRx \Rightarrow x = y$.

A relation R **on** a set X is a relation with $\mathrm{dom}(R), \mathrm{rg}(R) \subseteq X$. A relation R on X is called

- **reflexive** if $\forall\, x \in X : xRx$
- **total** if $\forall\, x, y \in X : xRy \vee yRx$.

A relation R on a set X is a

- **quasi-order** if R is reflexive and transitive
- **partial order** if R is an antisymmetric quasi-order
- **linear order** if R is a total partial order.

Quasi-ordered sets

A pair (X, R) with R a quasi-order, partial order or linear order is respectively a **quasi-ordered**, **partial ordered** or **linearly ordered set** ("qoset", "poset", "loset").

Quasi-orders are often denoted by "$\leq$".

Using $\mathrm{id}_X := \{(x, x) : x \in X\}$ for the identity map on the set X, we have for every partial order R on X

$$\mathrm{id}_X \subseteq R \subseteq X^2,$$

where

- (X, id_X) is the **discrete order** for X (any two (different) elements are incomparable)
- (X, X^2) is the **indiscrete order** for X (any two elements are equivalent).

Preordered categories

A category $\mathfrak{C}$ is called **preordered** if for all $X, Y \in \mathrm{Obj}(\mathfrak{C})$ there is at most one morphisms from X to Y.

- Given a quasi-ordered set $\underline{X} = (X, \leq)$, we obtain a preordered category $\mathrm{cat}(\underline{X})$ via
  - the object set is X;
  - the morphism set is $\leq$;
  - for a morphism $f = (x, y)$ we set $\mathrm{dom}(f) := x$ and $\mathrm{cod}(f) := y$;
  - the composition of morphisms (x, y) and (y, z) is (x, z);
  - the identity of object x is (x, x).
- Given any category $\mathfrak{C}$, we obtain a quasi-ordered set $\mathrm{qos}(\mathfrak{C}) := (\mathrm{Obj}(\mathfrak{C}), \leq)$ by

$$X \leq Y \Leftrightarrow \exists f \in \mathrm{Mor}(\mathfrak{C}) : \mathrm{dom}(f) = X \wedge \mathrm{cod}(f) = Y.$$

Again "essentially" the same

Preordered categories and quasi-ordered sets are "essentially" the same in the following sense:

1. Given a quasi-ordered set X we have $\mathrm{qos}(\mathrm{cat}(X)) = X$.
2. Given a preordered category $\mathfrak{C}$, the category $\mathrm{cat}(\mathrm{qos}(\mathfrak{C}))$ is the same as $\mathfrak{C}$ except of that the morphism names have been lost.

Special preordered categories are **discrete categories**, which have no morphisms except of the identity morphisms:

1. Discrete categories correspond to discrete orders.
2. Discrete categories can be thought of just as (bare) sets.

Refinements

As preordered categories correspond to qosets, we have furthermore:

1. Posets correspond to preordered categories, where the existence of a morphism from X to Y for $X \neq Y$ excludes the existence of a morphism from Y to X (such categories might be called "ordered categories").
2. Losets correspond to preordered categories, where for two different objects X, Y either there is a morphism from X to Y or a morphism from Y to X (but not both; such categories might be called "linearly ordered categories").

Since discrete orders are posets, discrete categories are the extreme cases of ordered categories.

Morphisms for directed general graphs

Since dgg's are (multi-sorted) algebraic structures, we inherit a natural notion of homomorphisms from (universal) algebra:

For dgg's $G = (V, E, \alpha, \beta)$ and $G' = (V', E', \alpha', \beta')$ a **homomorphism** $f : G \to G'$ from G to G' is a pair $f = (f_V, f_E)$ of maps

$$f_V : V \to V', \quad f_E : E \to E',$$

such that for all $e \in E$ we have

$$\begin{aligned} f_V(\alpha(e)) &= \alpha'(f_E(e)) \\ f_V(\beta(e)) &= \beta'(f_E(e)). \end{aligned}$$

We call f_V the "vertex-map" of f, and f_E the "edge-map".

Basic properties of graph homomorphisms

1. For a dgg G we have $\mathrm{id}_G : G \to G$ for $\mathrm{id}_G := (\mathrm{id}_{V(G)}, \mathrm{id}_{E(G)})$.
2. If $f : G \to G'$ and $g : G' \to G''$, then $g \circ f : G \to G''$ for $g \circ f := (g_V \circ f_V, g_E \circ f_E)$.

If $f : G \to G'$, and for some $x \in V(G) \cup E(G)$ it is clear from the context whether it is a vertex or an edge of G, then we just write "$f(x)$" for the application of the vertex-map resp. the edge-map associated with f.

Remarks: In NP

Graph homomorphisms (typically for finite or countable graphs) have received steadily growing attention in graph theory, constraint satisfaction theory and complexity theory, since the question whether there exists a homomorphisms $f : G \to G'$ for some fixed G' generalises the **graph colouring problem** (and is itself NP-complete for "most G'").

Thus graph homomorphisms seem to have a distinctively different flavour than morphisms in algebra and topology, where never the mere existence of a morphism is the fundamental problem (but only question about the existence of *special* morphisms like embeddings or isomorphisms). Accordingly the study of the category of graphs has a "combinatorial flavour".

Definition of a functor

A **functor** $F : \mathfrak{C} \to \mathfrak{D}$ from a category $\mathfrak{C}$ to a category $\mathfrak{D}$ is a dgg-homomorphism $F : \text{dgg}(\mathfrak{C}) \to \text{dgg}(\mathfrak{D})$ such that

1. for all $X \in \text{Obj}(\mathfrak{C})$ we have $F(\text{id}(\mathfrak{C})_X) = \text{id}(\mathfrak{D})_{F(X)}$;
2. for all $(f, g) \in W_2'(\mathfrak{C})$ we have $F(g \circ_{\mathfrak{C}} f) = F(g) \circ_{\mathfrak{D}} F(f)$.

The vertex-map of F is now called the "object-map", and the edge-map the "morphism-map".

The two basic properties of functors are (using categories $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$):

- $\text{id}_{\mathfrak{C}}$ is a functor from $\mathfrak{C}$ to $\mathfrak{C}$, where $\text{id}_{\mathfrak{C}} := \text{id}_{\text{dgg}(\mathfrak{C})}$.
- If $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \to \mathfrak{E}$, then $G \circ F : \mathfrak{C} \to \mathfrak{E}$.

Graph homomorphisms yield functors

Consider dgg's G, G' and a homomorphism $f : G \to G'$. We obtain a functor

$$\mathbf{cat}(\boldsymbol{f}) : \mathrm{cat}(G) \to \mathrm{cat}(G')$$

between the associated free categories as follows:

- The object-map of $\mathrm{cat}(f)$ is the same as the vertex-map of f.
- For a morphism of $\mathrm{cat}(G)$, that is, a walk $v_0, \dots, e_m, v_m$ in G, we define

  $$\mathrm{cat}(f)(v_0, \dots, e_m, v_m) := f(v_0), \dots, f(e_m), f(v_m)$$

  (yielding a walk in G', that is, a morphism of $\mathrm{cat}(G')$).

Monoid-homomorphisms

Given monoids $(M, \circ, e)$, $(M', \circ', e')$, a **homomorphism** $f : (M, \circ, e) \to (M', \circ', e')$ is a map $f : M \to M'$ such that

$$f(e) = e'$$
$$f(x \circ y) = f(x) \circ' f(y)$$

for all $x, y \in M$.

We see that homomorphisms between monoids are "essentially" the same as functors between one-object-categories.

Monoid-homomorphisms continued

More precisely:

- Given a homomorphism $f : (M, \circ, e) \to (M', \circ', e')$, we obtain a functor

$$\mathrm{cat}(f) : \mathrm{cat}((M, \circ, e)) \to \mathrm{cat}((M', \circ', e'))$$

  by using f for the morphism-map of $\mathrm{cat}(f)$, while there is only one object-map.
- Given a functor from one one-object category $\mathfrak{C}$ to another one-object category $\mathfrak{D}$, the morphism-map is a homomorphism of the associated monoids.

Order-homomorphisms

Given qosets $(M, \leq)$, $(M', \leq')$, a **homomorphism** $f : (M, \leq) \to (M', \leq')$ is a map $f : M \to M'$ such that

$$x \leq y \Rightarrow f(x) \leq' f(y)$$

for all $x, y \in M$.

We see that homomorphisms between qosets are "essentially" the same as functors between preordered categories.

Order-homomorphisms continued

More precisely:

- Given a homomorphism $f : (M, \leq) \to (M', \leq')$, we obtain a functor

$$\mathrm{cat}(f) : \mathrm{cat}((M, \leq)) \to \mathrm{cat}((M', \leq'))$$

  by using f for the object-map of $\mathrm{cat}(F)$, while for a morphism of $\mathrm{cat}((M, \leq))$, that is, a pair $(x, y) \in \leq$, we define $f((x, y)) := (f(x), f(y))$.
- Given a functor $F : \mathfrak{C} \to \mathfrak{D}$ between (arbitrary) categories, we obtain a homomorphism

$$\mathrm{qos}(F) : \mathrm{qos}(\mathfrak{C}) \to \mathrm{qos}(\mathfrak{C})$$

  by taking the object-map of F.

Why is everything so small??

Yet we have seen

1. the empty category
2. free categories (from dgg's)
3. one-object categories (aka monoids)
4. "thin categories" (preordered categories, aka quasi-ordered sets).

Furthermore

1. we lifted graph homomorphisms to functors between free categories
2. interpreted functors between one-object categories as monoid-homomorphisms
3. interpreted functors between preordered categories as quasi-ordered-set-homomorphisms.

Is that everything?!? Now we are grown up, ready to leave our small categories, and hungry to conquer the large categories.

No universal sets

By Russell's paradox we get

$$\forall x \, \exists y : y \notin x.$$

By variation on this principle, there is

- no set of all sets
- no set of all finite sets
- no set of all monoids
- no set of all qosets

and so on. We solve this problem by first modelling such "large categories" by "meta-categories", and in a second step we restrict meta-categories to appropriate "very large sets".

Predicates and operations

We need the following *meta-concepts*:

- An n**-ary predicate** for $n \in \mathbb{N}$ is a formula $P(x_1, \ldots, x_n)$ of set theory.
- An n**-ary operation** for $n \in \mathbb{N}$ is an $(n+1)$-ary predicate $O(x_1, \ldots, x_n, y)$ such that $\forall\, x_1, \ldots, x_n\, \exists!\, y : O(x_1, \ldots, x_n, y)$ is a valid sentence of set theory; $O(x_1, \ldots, x_n)$ is used as abbreviation for this unique y.

The notion of meta-categories

A **meta-category** is a sixtuple (O, M, d, c, C, i) (at the meta-level), where O, M are unary predicates, C is a binary operation and d, c, i are unary operations such that

(i) $\forall f : M(f) \to O(d(f)) \wedge O(c(f))$

(ii) $\forall f, g : M(f) \wedge M(g) \wedge c(f) = d(g) \to$
$M(C(f, g)) \wedge d(C(f, g)) = d(f) \wedge c(C(f, g)) = c(g)$

(iii) $\forall x : M(I(x)) \wedge d(I(x)) = c(I(x)) = x$

(iv) $\forall f, g, h : M(f) \wedge M(g) \wedge M(h) \wedge$
$c(f) = d(g) \wedge c(g) = d(h) \to$
$C(f, C(g, h)) = C(C(f, g), h)$

(v) $\forall x, f, g : M(f) \wedge M(g) \wedge O(x) \wedge$
$d(f) = x \wedge c(g) = x \to$
$C(i(x), f) = f \wedge C(g, i(x)) = g$.

Fitting into sets

A meta-category (O, M, d, c, C, i) **fits** into a set $\mathbb{U}$ if

$$\forall f : M(f) \wedge d(f) \in \mathbb{U} \wedge c(f) \in \mathbb{U} \rightarrow f \in \mathbb{U}$$

is true.

If now (O, M, d, c, C, i) fits into $\mathbb{U}$, then the **induced category** $(O, M, d, c, C, i)_{\mathbb{U}}$ is given as follows:

1. the object set is the set of $x \in \mathbb{U}$ with $O(x)$;
2. the morphism set is the set of f with $M(f)$ such that $d(f)$ and $c(f)$ are in the object set;
3. domain, codomain, composition and identity are the respective operations restricted to the resp. stripped sets of walks (the *replacement scheme* of set theory guarantees that we get four maps in this way).

What size?

Every meta-category fits into the empty set. This is quite reassuring. However we already know the resulting category.

We are looking for sets $\mathbb{U}$ such that

- the meta-category "naturally" fits into $\mathbb{U}$
- and the resulting category does not "really" depend on $\mathbb{U}$.

In other words, $\mathbb{U}$ should be large enough

1. such that all interesting objects are in it,
2. and by normal means we cannot get out of it.

The notion of a "(set theoretic) universe" offers a good (universal(!)) solution for this problem.

The notion of a set-theoretical universe

Set theory for us (and for the moment) means ZFC (Zermelo-Fraenkel with the axiom of choice). A **universe** now is a set $\mathbb{U}$ which itself yields a model of ZFC.

More specifically, a set $\mathbb{U}$ is a universe iff the following (not independent) conditions are fulfilled:

(i) $\forall x \in \mathbb{U} \, \forall y : (y \in x \vee y \subseteq x) \rightarrow y \in \mathbb{U}$

(ii) $\forall x, y \in \mathbb{U} : \{x, y\} \in \mathbb{U}$

(iii) $\forall f : A \rightarrow B : A, B \in \mathbb{U} \rightarrow f \in \mathbb{U}$

(iv) $\forall A, B \in \mathbb{U} \, \forall f : A \rightarrow B : \bigcup_{a \in A} f(a), \prod_{a \in A} f(a) \in \mathbb{U}$

(v) $\forall A \in \mathbb{U} : \mathbb{P}(A) \in \mathbb{U}$

(vi) There exists an infinite set in $\mathbb{U}$.

(Recall: $\mathbb{P}(A) = \{T : T \subseteq A\}$ and $\bigcup_{a \in A} f(a) = \{x \mid \exists\, a \in A : x \in f(a)\}$.)

The universes axiom

We consider **ZFCU** as the appropriate foundations of mathematics, which adds the (strong) universes axiom to ZFC:

For every set x there exists a universe $\mathbb{U}$ with $x \in \mathbb{U}$.

(This axiom is equivalent to some axiom for inaccessible cardinals; unfortunately I don't have a good source for this yet.)

By the universes axiom for every set x there is a unique smallest (w.r.t. subsumption) universe $\mathbb{U}$ containing x, which we call **the universe of** x.

(Note that $\{\mathbb{U}\} \notin \mathbb{U}$, though $\{\mathbb{U}\}$ just contains one element.)

Universes-fitting

We call a meta-category **universes-fitting** if it fits into every universe:

- I'm not aware of any (useful) category which is not universes-fitting.
- I'm not aware of literature here, but it should be possible to prove, that for universes-fitting meta-categories the size of the universe does not "really matter" for the induced category.

A universe represents a closed world (it is the closure of some set under "all" set-theoretic operations):

1. When speaking of the category of "all sets" or "all categories", we are not really interested in "really all", but in a formation which is stable under all reasonable operations.
2. Every external category of "really all sets" or "really all categories" can be modelled by just choosing an appropriate universe.

Correspondences

A **correspondence** is a triple (R, A, B), where A, B are sets (the **domain** and **codomain** of the correspondence) and $R \subseteq A \times B$.

Correspondences can be viewed as "partial set-valued maps" (from A to B).

For correspondences (R, A, B) and (S, B, C) we define the **composition**

$$\boldsymbol{(S, B, C) \circ (R, A, B)} := (S \circ R, A, C)$$

(which again is a correspondence).

Properties of correspondences

A correspondence (R, A, B) is

- **left-unique** resp. **right-unique** if R is;
- **left-total** resp. **right-total** if $\text{dom}(R) = A$ resp. $\text{cod}(R) = B$.

A **map (as triple)** is a left-total right-unique correspondence.

Very often there is the “bare” notion of a morphism, which does not allow to infer domain and codomain (structures(!)), and then the corresponding “triple version” simply includes domain and codomain structures.

The category of correspondences

The **meta-category of correspondences** is given by

- objects are sets;
- morphisms are correspondences;
- domains and codomains are given as defined for correspondences;
- composition is the composition of correspondences;
- the identities are the identity maps.

For any universe $\mathbb{U}$ the induced category of correspondences is denoted by $\mathfrak{KOR}_{\mathbb{U}}$ (where typically the universe is dropped).

$2^4 = 16$ categories of sets

If σ denotes any of the 2^4 combinations of the properties "lu" (left-unique), "ru" (right-unique), "lt" (left-total) or "rt" (right-total), then we obtain a partial category $\mathfrak{KOR}_\sigma$ by considering only σ-correspondences (the proof of this is one of the many exercises). Especially:

1. $\mathfrak{KOR}$ is the **category of partial set-valued maps.**
2. $\mathfrak{SET} := \mathfrak{KOR}_{\mathrm{ru,lt}}$ is the **category of sets.**
3. $\mathfrak{KOR}_{\mathrm{ru}}$ is the **category of partial maps.**
4. $\mathfrak{KOR}_{\mathrm{lt}}$ is the **category of multivalued maps.**
5. $\mathfrak{KOR}_{\mathrm{lu,ru,lt}}$ is the **category of injective maps.**
6. $\mathfrak{KOR}_{\mathrm{ru,lt,rt}}$ is the **category of surjective maps.**
7. $\mathfrak{KOR}_{\mathrm{lu,ru,lt,rt}}$ is the **category of bijective maps.**

Concrete categories

Categories (or meta-categories)

- whose objects consist of "structured sets", i.e., there exists a $k \in \mathbb{N}$ such that all objects are k-tuples t with first component a set t_1,
- where the morphisms from t to t' are (special) maps from t_1 to t'_1,
- and where composition is composition of maps,
- and identities are the identical maps,

are called **concrete categories** (preliminary definition).

The first example of a concrete category is $\mathfrak{SET}$.

Given a concrete meta-category, when forming the induced concrete category, and the notion of morphism employed does not allow to infer domain and target objects, automatically the *triple construction* is performed.

Further convention

- When defining concrete meta-categories, only the objects and the morphisms need to be mentioned.
- And the construction of (universes-fitting(!)) meta-categories as a first step is automatically implicitly performed when defining a category by notions like “all spaces”.

The categories $\mathfrak{SET}_{\mathbb{U}}$ for example are defined in this way by suppressing the universe-index (which then becomes free for other purposes), and just saying:

$\mathfrak{SET}$ is the (concrete) category of all sets, where the morphisms are all maps between them.

Two concrete and one "biconcrete" categories

- The concrete category $\mathfrak{MON}$ of monoids has as objects all monoids, while the morphisms are the monoid homomorphisms.
- The concrete category $\mathfrak{QORD}$ of quasi-ordered sets has as objects all quasi-ordered sets, while the morphisms are the homomorphisms of quasi-ordered sets.

The category $\mathfrak{DGG}$ of dgg's has as objects all dgg's, morphisms are homomorphisms of dgg's, and composition of dgg's and identities are as defined.

$\mathfrak{DGG}$ is not a concrete category, but it looks quite close to a concrete category: $\mathfrak{DGG}$ is not a concrete category "over $\mathfrak{SET}$", but "over $\mathfrak{SET}^2$" (as will be defined later).

The category of categories

- The category $\mathfrak{CAT}$ of all categories has objects all categories, morphisms are functors between categories, and composition and identities are as defined.
- Like $\mathfrak{DGG}$, the category $\mathfrak{CAT}$ is not a concrete category "over $\mathfrak{SET}$" but "over $\mathfrak{SET}^2$".

We have identified two interesting induced (full) subcategories of $\mathfrak{CAT}$:

1. The full subcategory $\mathfrak{MON}'$ of all one-object categories, which is "essentially" the same as $\mathfrak{MON}$.
2. The full subcategory $\mathfrak{QORD}'$ of all preordered (thin) categories, which is "essentially" the same as $\mathfrak{QORD}$.

Remarks: "Essentially" ?!

I leave it as an exercise (?) to find out the exact relationship between $\mathfrak{MON}'$ and $\mathfrak{MON}$, and between $\mathfrak{QORD}'$ and $\mathfrak{QORD}$:

- The functor $\mathrm{cat} : \mathfrak{MON} \to \mathfrak{CAT}$ is "quite close" to an isomorphism (when restricting the image). For a universe $\mathbb{U}$ one should be able to "construct" actually an isomorphism between $\mathfrak{MON}'_{\mathbb{U}}$ and $\mathfrak{MON}_{\mathbb{U}}$. (As a start one could use the natural isomorphism between $\mathfrak{MON}'$ and $\mathfrak{MON} \times \mathrm{cat}((\mathbb{U}, \mathbb{U}^2))$ (the latter is the "complete" preordered category over $\mathbb{U}$).)
- The functor $\mathrm{cat} : \mathfrak{QORD} \to \mathfrak{CAT}$ is "quite close" to an isomorphism (image-restricted). For a universe $\mathbb{U}$ one should be able to "construct" actually an isomorphism between $\mathfrak{QORD}'_{\mathbb{U}}$ and $\mathfrak{QORD}_{\mathbb{U}}$.

Functors we have seen so far

Among the functors we have "seen" (please check(!)), we have the following functors into the category of categories (the three basic constructions for small categories):

1. cat : $\mathfrak{DGG} \to \mathfrak{CAT}$ (the formation of free categories over a dgg).
2. cat : $\mathfrak{MON} \to \mathfrak{CAT}$ (interpreting monoids as one-object categories).
3. cat : $\mathfrak{QORD} \to \mathfrak{CAT}$ (interpreting quasi-ordered sets as thin categories).

And we have "seen" the functors dgg : $\mathfrak{CAT} \to \mathfrak{DGG}$ and qos : $\mathfrak{CAT} \to \mathfrak{QORD}$ which *forget some structure*.

Now what about the functors defined on concrete categories like $\mathfrak{MON}$ and $\mathfrak{QORD}$, which simply forget *all* the special structure, and uncover bare sets and maps?

The discourse-universe

Before coming to forgetful functors, we should discuss the hidden indices in a notation like

$$\text{“cat} : \mathfrak{MON} \to \mathfrak{CAT}\text{”},$$

since actually both categories are only defined “up to a universe”:

When relating two universe-indexed categories $\mathfrak{C}_{\mathbb{U}}$ and $\mathfrak{D}_{\mathbb{U}'}$, then we assume $\mathbb{U} = \mathbb{U}'$, otherwise the relation between the universes should be discussed.

Sizing categories

- We define the universe $\mathbb{U}(\mathfrak{C})$ of a category as the smallest universe $\mathbb{U}$ with $\mathrm{Obj}(\mathfrak{C}), \mathrm{Mor}(\mathfrak{C}) \subseteq \mathbb{U}$.
- Note that, although $\mathrm{card}(\mathrm{Obj}(\mathfrak{C})) \le \mathrm{card}(\mathrm{Mor}(\mathfrak{C}))$, since $\mathrm{id}_{\mathfrak{C}} : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Mor}(\mathfrak{C})$ is injective, it could $\mathrm{Obj}(\mathfrak{C})$ nevertheless require a higher universe than $\mathrm{Mor}(\mathfrak{C})$ due to "representational excess".
  The two conditions $\mathrm{Obj}(\mathfrak{C}), \mathrm{Mor}(\mathfrak{C}) \subseteq \mathbb{U}$ are equivalent to the single condition $\circ(\mathfrak{C}) \subseteq \mathbb{U}$.
- For a category $\mathfrak{C}_{\mathbb{U}}$ induced by a meta-category on a universe $\mathbb{U}$ we have $\mathbb{U}(\mathfrak{C}_{\mathbb{U}}) \subseteq \mathbb{U}$, and normally we have equality here, but strict subsumption (in which case we actually have $\mathbb{U}(\mathfrak{C}_{\mathbb{U}}) \in \mathbb{U}$) could occur if the meta-category is not "powerful" enough to fill the space $\mathbb{U}$ given to it.

“Small” and “large” categories

A category $\mathfrak{C}$ is called **small** if $\mathfrak{C} \in \mathbb{U}(\mathfrak{C})$, and **large** otherwise:

1. Note that “small” and “large” are relative notions here, so that for example a sub-category of a small category could be a large category (in its own, smaller, universe).
2. Also a small category can be much larger (in terms of cardinality) than a large category.
3. However, when fixing the universe of discourse, than “small” and “large” really refer to qualitative differences in size.

The distinction between small and large categories corresponds to the distinction between “sets and classes” (but at many different levels). For us, the notions “small” and “large” are mainly used to connect to other literature, and for illustrational purposes.

Forgetful functors

Now for a concrete category $\mathfrak{C}$ we define the **forgetful functor** $V : \mathfrak{C} \to \mathfrak{SET}$, which strips away the structure and leaves the bare sets and maps, more precisely as

$$V : \mathfrak{C} \to \mathfrak{SET}_{\mathbb{U}(\mathfrak{C})}.$$

Thus we get the forgetful functors $V : \mathfrak{MON} \to \mathfrak{SET}$ and $V : \mathfrak{QORD} \to \mathfrak{SET}$. Some remarks:

1. There could be some "excess" in the choice of the universe for $\mathfrak{SET}$ in the case that the structure for a set does not fit into the same universe as the set itself. This does not occur in natural examples.
2. The definition of forgetful functors for concrete categories $\mathfrak{C}$ overlaps with the convention for the discourse-universe in case $\mathfrak{C}$ is meta-category induced: Both treatments agree in all natural cases.

Interlude: Product categories

Given a family $(\mathfrak{C}_i)_{i \in I}$ of categories, the **product category**

$$\prod_{i \in I} \mathfrak{C}_i$$

is defined as follows:

1. The object set is $\prod_{i \in I} \mathrm{Obj}(\mathfrak{C}_i)$.
2. The morphism set is $\prod_{i \in I} \mathrm{Mor}(\mathfrak{C}_i)$.
3. Composition of morphisms happens componentwise.
4. Identity morphisms are the componentwise-identities.

As for sets, the product category $\mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n$ for $n \in \mathbb{N}$ uses tuples (instead of families) for the representation of objects and morphisms.

The relation between $\mathfrak{CAT}$ and $\mathfrak{DGG}$

As we have already remarked, we have a forgetful functor

$$V_{\mathfrak{DGG}} : \mathfrak{DGG} \to \mathfrak{SET}^2$$

(forgetting the source and target map), and in the same vein we have

$$V_{\mathfrak{CAT}} : \mathfrak{CAT} \to \mathfrak{SET}^2.$$

Now actually the latter forgetful functor should be factorised as

$$V_{\mathfrak{CAT}} = V_{\mathfrak{DGG}} \circ \mathrm{dgg}_{\mathfrak{CAT}},$$

using the forgetful functor

$$\mathrm{dgg}_{\mathfrak{CAT}} : \mathfrak{CAT} \to \mathfrak{DGG}$$

which extracts the underlying dgg from a category.

Outlook: "Adjoint pairs"

Recall the pair $(\mathrm{cat}_{\mathfrak{DGG}}, \mathrm{dgg}_{\mathfrak{CAT}})$ of reverse functors

$$\mathrm{cat}_{\mathfrak{DGG}} : \mathfrak{DGG} \to \mathfrak{CAT}, \quad \mathrm{dgg}_{\mathfrak{CAT}} : \mathfrak{CAT} \to \mathfrak{DGG}.$$

The functor cat here yields the "free category of a dgg", and the pair $(\mathrm{cat}_{\mathfrak{DGG}}, \mathrm{dgg}_{\mathfrak{CAT}})$ is called an "adjoint pair".

Another adjoint pair is $(\mathrm{qos}_{\mathfrak{CAT}}, \mathrm{cat}_{\mathfrak{QORD}})$ for $\mathrm{qos}_{\mathfrak{CAT}} : \mathfrak{CAT} \to \mathfrak{QORD}$ and $\mathrm{cat}_{\mathfrak{QORD}} : \mathfrak{QORD} \to \mathfrak{CAT}$ as considered before.

In the same spirit, formation of the free monoid A^* over a set A is "left-adjoint" to $V : \mathfrak{MON} \to \mathfrak{SET}$.

(As a (very) informal definition for now, the left-adjoint (if it exists(!)) of a forgetful functor yields "free objects", the most general way of adding structure with excess.)

Exercises

1. Does the forgetful functor $V_{\mathfrak{CAT}} : \mathfrak{CAT} \to \mathfrak{SET}^2$ has a left-adjoint (the "free way" of creating a category, given a set of objects and a set of morphisms)?
2. Perhaps we should first ask for the left-adjoint of $V_{\mathfrak{DGG}} : \mathfrak{DGG} \to \mathfrak{SET}^2$, and then combine the construction of this "free dgg" from a pair of sets with the construction of the free category from a dgg, obtaining the free category from a pair of sets?!

One of the beautiful things about category theory is "dualisation": We can also ask about the existence of "right-adjoints" of forgetful functors! (Asking for the "cofree objects", like indiscrete spaces.)

The dual category

1. For a dgg $G = (V, E, \alpha, \beta)$ we define the **transposed dgg G^t** as
$$G^t := (V, E, \beta, \alpha).$$
2. For a category $\mathfrak{C} = (G, \circ, \mathrm{id})$ we define the **dual category $\mathfrak{C}^t$** as
$$\mathfrak{C}^t := (G^t, \circ^t, \mathrm{id}),$$
where $\circ^t((f, g)) = \circ(g, f)$.

It is not hard to see that $\mathfrak{C}^t$ actually is also a category (with the same object and morphisms sets and the same identities — only the arrow directions are reversed). We have
$$(\mathfrak{C}^t)^t = \mathfrak{C}.$$

Remark on categories of correspondences

Given a correspondency-type σ, we define the "dual" σ^{t} by flipping "left" and "right".

Now the category $\mathfrak{KOR}_{\sigma}^{\mathrm{t}}$ is isomorphic to the category $\mathfrak{KOR}_{\sigma^{\mathrm{t}}}$, using the identity on the object set, and formation of the inverse correspondence on the morphism set:

$$(R, A, B)^{-1} := (R^{-1}, B, A)$$
$$R^{-1} := \{(y, x) : (x, y) \in R\}.$$

Contravariant functors

Considering functors in the simple sense (not as triples), for arbitrary categories $\mathfrak{C}, \mathfrak{D}$ we have the equivalences

$$F : \mathfrak{C} \to \mathfrak{D} \Leftrightarrow F : \mathfrak{C}^{\mathrm{t}} \to \mathfrak{D}^{\mathrm{t}}$$
$$F : \mathfrak{C}^{\mathrm{t}} \to \mathfrak{D} \Leftrightarrow F : \mathfrak{C} \to \mathfrak{D}^{\mathrm{t}}.$$

In the first case F is called a **covariant functor**, in the second case a **contravariant functor.**

(We also see that duality of categories yields an automorphism of the category of all categories.)

The Hom-functor in two arguments

For a category $\mathfrak{C}$ let $\mathbf{Mor}_{\mathfrak{C}}(-,-)$ be the map with domain $\mathrm{Obj}(\mathfrak{C})^2$ given by

$$\mathrm{Mor}_{\mathfrak{C}}(X,Y) := \{f \in \mathrm{Mor}(\mathfrak{C}) : \mathrm{dom}(f) = X \wedge \mathrm{cod}(f) = Y\}$$

(one also find the notation "$\mathfrak{C}(X,Y) := \mathrm{Mor}_{\mathfrak{C}}(X,Y)$", but in this way we cannot easily refer to the whole map).

If the range of $\mathrm{Mor}_{\mathfrak{C}}$ is a subset of $\mathbb{U}(\mathfrak{C})$, then let $\mathbb{U} := \mathbb{U}(\mathfrak{C})$, while otherwise let $\mathbb{U}$ be the universe of $\mathbb{U}(\mathfrak{C})$. Now we have functor

$$\mathrm{Mor}_{\mathfrak{C}}(-,-) : \mathfrak{C}^{\mathrm{t}} \times \mathfrak{C} \to \mathfrak{SET}_{\mathbb{U}}.$$

Exercise (and Talk 2) to specify the morphism map of this "bi-hom-functor".

Definition of a natural transformation

Given categories $\mathfrak{C}, \mathfrak{D}$ and functors $F, G : \mathfrak{C} \to \mathfrak{D}$, a **natural transformation** $\eta : F \to G$ is a family $\eta = (\eta_X)_{X \in \mathrm{Obj}(\mathfrak{C})}$ in $\mathrm{Mor}(\mathfrak{D})$ (in other words, a map $\eta : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Mor}(\mathfrak{D})$, but using family-notation) with $\eta_X : F(X) \to G(X)$ such that for all objects A, B in $\mathfrak{C}$ and morphisms $f : A \to B$ we have

$$\eta_B \circ F(f) = G(f) \circ \eta_A.$$

In other words, the diagrams

$$\begin{array}{ccc} G(A) & \xrightarrow{G(f)} & G(B) \\ \uparrow{\scriptstyle \eta_A} & & \uparrow{\scriptstyle \eta_B} \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

commute.

Remark: Commutative diagrams

1. A **diagram** in a category $\mathfrak{C}$ is a dgg-homomorphism $D : G \to \mathrm{dgg}(\mathfrak{C})$ for some dgg G.
   The dgg G is called here the "diagram scheme".
2. Because of freeness, there is exactly one extension

   $$D' : \mathrm{cat}(G) \to \mathfrak{C}.$$

   (This is essentially the same as lifting a dgg-homomorphism $f : G \to G'$ to a functor $\mathrm{cat}(f) : \mathrm{cat}(G) \to \mathrm{cat}(G')$.)
3. Now D is **commutative** iff D' is constant on every morphism set $\mathrm{Mor}_{\mathrm{cat}(G)}(u, w)$ in $\mathrm{cat}(G)$.

Remark: Commutativity and preorder

1. A functor $F : \mathfrak{C} \to \mathfrak{D}$ induces a **congruence relation** $\sim_F$ on $\mathfrak{C}$ via

$$f \sim_F g :\Longleftrightarrow \operatorname{dom}(f) = \operatorname{dom}(g) \wedge \operatorname{cod}(f) = \operatorname{cod}(g) \wedge F(f) = F(g)$$

   for morphisms $f, g \in \operatorname{Mor}(\operatorname{Obj}(\mathfrak{C}))$.
2. For every congruence relation $\sim$ on a category $\mathfrak{C}$ we obtain the **quotient category** $\mathfrak{C}/\sim$ in the usual way (keeping all objects, while replacing morphisms by equivalence classes of morphisms).
3. Now a diagram $D : G \to \operatorname{dgg}(\mathfrak{C})$ is commutative iff $\operatorname{cat}(G)/\sim_{D'}$ is a preordered category.

Categories of functors

1. For every functor $F : \mathfrak{C} \to \mathfrak{D}$ we have the identity transformation $\mathrm{id}_F : F \to F$ given by $(\mathrm{id}_{F(X)})_{X \in \mathrm{Obj}(\mathfrak{C})}$.
2. Consider functors $F, G, H : \mathfrak{C} \to \mathfrak{D}$. For natural transformations $\eta : F \to G$ and $\zeta : G \to H$ the composition
$$\zeta \circ \eta : F \to H$$
is given by
$$(\zeta \circ \eta)_X := \zeta_X \circ \eta_X.$$

Thus for categories $\mathfrak{C}, \mathfrak{D}$ we obtain the **functor category**

$$\mathfrak{FUN}(\mathfrak{C}, \mathfrak{D}),$$

which has as objects the functors from $\mathfrak{C}$ to $\mathfrak{D}$, and as morphisms the natural transformations between functors (together with the above composition of natural transformation and the identity transformations).

Entry exam into the world of categories

$$\mathfrak{FUN} : \mathfrak{CAT}^{t} \times \mathfrak{CAT} \to \mathfrak{CAT}.$$

```
Provide the missing definitions,
and prove that we have a functor.
```

The size of index sets

In a formulation like

"Let $(A_i)_{i \in I}$ be a family of objects in the category $\mathfrak{C}$.",

the set I is just an *arbitrary* set. If we don't want arbitrary sets, we have to explicitely say so (for example, we speak of "finite families").

However, there are situations where we implicitly assume a certain natural restriction on the size of the index sets:

- "The category of sets has all products." (Somehow asserting, that for families $(X_i)_{i \in I}$ in $\mathfrak{SET}$ also $\prod_{i \in I} X_i$ is an object of $\mathfrak{SET}$.)
- "The category of categories has all products." (Somehow asserting, that for families $(\mathfrak{C}_i)_{i \in I}$ in $\mathfrak{CAT}$ also $\prod_{i \in I} \mathfrak{C}_i$ is an object of $\mathfrak{CAT}$.)

For arbitrary I, these statements are *always false*.

The “normal size” of index sets

The problem is, that the size of I has to be bounded (I represents the arity of infinitary operations as $\prod_{i \in I} X_i$).

For a category $\mathfrak{C}$, a **“normal” index set** (if we somehow assume such a notion) is an index set of the same cardinality as some element of $\mathbb{U}(\mathfrak{C})$, the universe of $\mathfrak{C}$.

Actually, in most situations it is sufficient to consider $I \in \mathbb{U}(\mathfrak{C})$ (like in statements of the form “all products exist”) — recall the literal meaning of “universe”.

$\mathbb{U}(\mathfrak{C})$ can be called the **indexing universe** of $\mathfrak{C}$.

However, there is an important exception, when we go to

“higher-order spaces” aka functor categories

(one of the strong points of category theory!).

A typical examples: “Weak presheaves”

Consider a small category $\mathfrak{C}$ and the category

$$\mathfrak{FUN}(\mathfrak{C}^{\mathrm{t}}, \mathfrak{SET})$$

of set-valued contravariant functors (if $\mathfrak{C}$ carries some appropriate extra structure, called a “coverage” (or, in older texts, a “Grothendiek pretopology”), then these functors are called “presheaves on $\mathfrak{C}$”).

According to the general convention, there is a universe $\mathbb{U}$ such that $\mathfrak{C} \in \mathbb{U}$ and $\mathfrak{SET} = \mathfrak{SET}_{\mathbb{U}}$. Now where do the “presheaves” live?!

“Presheaves” F should be small objects (i.e., $F \in \mathbb{U}$) — however, according to the triple construction, they are not! (Since $F = (F_0, \mathfrak{C}, \mathfrak{SET})$!)

The universe of $\mathfrak{FUN}(\mathfrak{C}^{\mathrm{t}}, \mathfrak{SET})$ is the universe $\mathbb{U}'$ containing $\mathbb{U}$ — which is far too big.

A general rule for this exception

So in this case the "indexing category" of $\mathfrak{FUN}(\mathfrak{C}^t, \mathfrak{SET})$ is again $\mathbb{U}$.

In general, for categories $\mathfrak{C}, \mathfrak{D}$:

1. If both $\mathfrak{C}, \mathfrak{D}$ are small (w.r.t. the same universe — this is always assumed in such contexts), then also $\mathfrak{FUN}(\mathfrak{C}, \mathfrak{D})$ is small, so there is no problem here.
2. If $\mathfrak{C}$ is small and $\mathfrak{D}$ is large, then the "indexing category" of $\mathfrak{FUN}(\mathfrak{C}, \mathfrak{D})$ is the current discourse-universe.
   And thus $\mathfrak{FUN}(\mathfrak{C}, \mathfrak{D})$ is considered as just a large category (not "super-large"), which can be justified by removing (in thoughts) the superfluous labels "$\mathfrak{D}$" of the functors (as triples).

(The whole problem consists just in "lazy formulations" like "all products" (or "all limits"), and one just needs to be a bit more precise here.)

Remark: Other reactions

The cause of the (small) problem here was the triple construction $(F, \mathfrak{C}, \mathfrak{D})$ for functors, which created a large object even for small $\mathfrak{C}$. In the literature one finds two alternative treatments:

- Abandon the triple construction: Instead of having "independent morphisms" one can have a framework ("categories as morphism-sets") where morphisms can only be accessed having domain and codomain at hand, and then the triple construction is no longer needed (set-theoretical ("naked") maps then for example are the morphisms of the category of sets).

Remark: Other reactions (continued)

- Ignore the object names: Extend the notion of “large categories”, and only ask for the morphism set to be a subset of the current universe. Every such “enlarged category” is isomorphic to a normal large category, and triples got harmless (they are just names, finally).

The second approach creates other trouble elsewhere, and has (apparently) not been further pursued in the literature, while the former has some advantages (and followers).

Further remark: “Small” and “large”

Traditionally, the distinction between small and large sets was the main tool to handle categories like the “category of categories” (sometimes “small” is called “set”, and “large” is called “class”). This method has some disadvantages:

- In order to keep things small certain hacks are used.
- At most places, the distinction is useless, but the distinction is floating around nevertheless.

In our (“modern”) framework the distinction between “small” and “large” is largely irrelevant:

We localised the problem by the use of “indexing universes”.

I think it’s better to mention problems explicitely, than to trust some kind of “semi-automatic” solution which causes more trouble than it solves.

Variations on the notion of a category

The main variations on the notion of a category occur (to the best of my knowledge) among the following (mainly orthogonal) dimensions:

I Different foundations are used (instead of ZFCU):

  (i) Other (weaker) set-theories are used.
  (ii) Or set-theory is avoided.

II Different basic structures are used (instead of dgg's):

  (i) The morphism-sets $\mathrm{Mor}(X, Y)$ are taken as basic.
  (ii) Or objects are dropped.

We will now discuss these variations, and argue that for our purpose (category theory as a mathematical theory like algebra or topology) the chosen foundations fits best.

The basic foundational issue

The basic problem is how to handle large categories like the category of “all sets”.

We use meta-categories for this purpose:

1. This is sufficient for us, since meta-categories are only devices to fill universes.
2. Without universes, meta-categories would not be suitable, since we want categories to be “first class citizens”, that is, objects of the theory, and not meta-constructions.
   (And not meta-meta-...-constructions(!)).

How to handle large categories in set theories without universes?

Using weaker set-theories

1. In ZFC there is no handle.
2. Some (older) texts uses "set theory with classes", which adds the notion of "(proper) classes" to set theory. However, this is just a hack: One can build the category of sets, and the category of "small categories", but these large categories now do not participate anymore in the "higher theory":

One of the revolutionary changes enabled by set theory is the possibility to go to more and more abstract spaces, on higher and higher levels — *ad infinitum* !
We should not go back to earlier stages of strictly separated "class societies".

(On top of classes one can stack super-classes ("conglomerates") and so on — all we get in this way is just a poor-mans version of ZFCU.)

Parameterising the logic

- A defensible quest is to search for minimal logical theories allowing to do category theory.
- This however is true for every mathematical subject, and it can get very complicated (and messy), and is furthermore a specialist's topic.
- For the general theory, we should go as far out as possible, without "foundational stinginess".
- Though some parts of category has been found intrinsically (?) linked with intuitionistic logic.

Comment: Not a tuple

Often categories are not "packed" into tuples, but a category is "somehow" an associations of several things:

1. This reflects the wish to keep even large categories "within the universe".
2. For a consistent set-theoretical treatment like ours, there is no need for such fuzziness.
3. We treat categories as *any other mathematical object*:
   1. The importance of small categories (as combinatorial objects, related to graph theory and constraint satisfaction) is on the rise.
   2. Yet for example "large groups" (of universe size) are only considered in set theory and model theory, but I expect also this to become more important over time.

   So the "traditional dichotomy", that categories are large and mathematical objects are small, is (slowly) melting down.

Starting with the morphism-sets

We have based the notion of a category on the notion of a directed general graph. A (popular) alternative starts with the morphism sets $\mathrm{Mor}_{\mathfrak{C}}(X, Y)$:

A "category via morphism-sets" is a quadruple

$$(O, (\mathrm{Mor}(X, Y))_{(X,Y)\in O^2}, (\circ_{X,Y,Z})_{(X,Y,Z)\in O^3}, (\mathrm{id}_X)_{X\in O}) :$$

1. O is a set (the set of objects);
2. $(\mathrm{Mor}(X, Y))_{(X,Y)\in O^2}$ is a family of sets (the family of morphism sets);
3. $(\circ_{X,Y,Z})_{(X,Y,Z)\in O^3}$ is a family of maps $\circ_{X,Y,Z} : \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ (the composition of morphisms);
4. $(\mathrm{id}_X)_{X\in O}$ is a family of morphisms $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ (the identity morphisms);

such that the composition of morphisms is associative and the identities are neutral elements.

Encodings

- From a category $\mathfrak{C}$ we obtain a "category via morphism-sets" $M(\mathfrak{C})$ by just using the morphism-sets $\mathrm{Mor}_{\mathfrak{C}}(X, Y)$ and the restrictions of the composition of morphisms.
- From a "category via morphism-sets" $\mathfrak{C}$ we obtain a category $m(\mathfrak{C})$ by using O as the object set,

$$\bigcup_{(X,Y) \in O^2} M(X, Y) \times \{X\} \times \{Y\}$$

as the morphism set (employing the triple construction), defining domain and codomain as the second resp. third component of a morphism (as triple), and combining all single "local" composition maps into one "global" composition.

Cryptomorphisms

The maps M and m translate the notion of category from one "language" into another (in the combinatorial world, where often no suitable notion of morphism is available, such maps are called "cryptomorphisms").

From a categorical point of view, let $\mathfrak{CAT}'$ be the category of "categories via morphism-sets":

1. $M : \mathfrak{CAT} \to \mathfrak{CAT}'$ and $m : \mathfrak{CAT}' \to \mathfrak{CAT}$ are "full embeddings" (full functors, which are injective on the morphisms (and objects)).
2. However, M and m are not isomorphisms, and the categories $\mathfrak{CAT}$ and $\mathfrak{CAT}'$ are only isomorphic via global maps which know about the universe (at least I assume so — left as exercise).

Disjoint morphism sets

Now for a category $\mathfrak{C}$ and $(X, Y), (X', Y') \in \mathrm{Obj}(\mathfrak{C})^2$ we actually have

$$(X, Y) \neq (X', Y') \Rightarrow \mathrm{Mor}_{\mathfrak{C}}(X, Y) \cap \mathrm{Mor}_{\mathfrak{C}}(X', Y') = \emptyset,$$

since otherwise from the morphisms (alone(!)) domain and codomain could not be recovered.

If thus we also demand disjointness for "categories via morphism-sets", we obtain "categories via disjoint morphism-sets":

1. Now for the cryptomorphism m we do not need to employ the triple construction anymore, and M and m become inverse to each other.
2. Thus, using $\mathfrak{CAT}''$ for the category of "categories via disjoint morphism-sets", we now have inverse isomorphisms M and m between $\mathfrak{CAT}$ and $\mathfrak{CAT}''$.

Disjoint or not?!

- "Categories via disjoint morphism-sets" have an easier "global structure" than without the disjointness-condition (they directly relate to directed general graphs).
- However, for "categories via morphisms-sets" we do never need to employ the triple constructions, and this makes it somewhat easier to handle morphisms (and functors) — that is, without hand waving.

Most texts which base the notion of categories on morphism-sets use the disjointness condition, however Borceaux in his "Handbook of categorical algebra" does not (avoiding in this way the little problem with functor categories).

Small morphism sets

Recall our definition of the Hom-bifunctor

$$\mathrm{Mor}_{\mathfrak{C}}(-_1, -_2) : \mathfrak{C}^{\mathrm{t}} \times \mathfrak{C} \to \mathfrak{SET}$$

Possibly the morphism sets $\mathrm{Mor}(X, Y)$ are large here. Now often for "categories via (disjoint) morphism-sets" the morphism-sets are required to be small:

- This requirement does not pose big problems,
- however it is unnecessary,
- and also somewhat unnatural from the point of view of "categories as dgg's".

Partial monoids

Ideologically speaking, the morphisms in a category are more important than the objects (which can be recovered from the identity morphisms). So there arises the wish to define "object-free" categories.

It is possible to define a category just as some form of "partial monoid", a triple

$$(M, \circ)$$

such that $\circ$ is a partial operation on M which is associative in the appropriate sense, and such that each element $f \in M$ has neutral elements "from the left and right".

Recovering the objects

1. It is a simple exercise to show that in a category for each object there is exactly one endomorphism which is left- and right-neutral regarding composition.
2. In the same vein, for a category as a partial monoid $(M, \circ)$ for every $f \in M$ there are unique neutral elements f_X, f_Y "from the left and right".
3. These special morphisms then identify the objects.

Categories as partial monoids are not used (at least not to my knowledge) — however for the field of partial semigroups the notions transferred/borrowed from category theory seem to be quite valuable.

The origins of category theory

End