# Representations of categories

Oliver Kullmann
Computer Science Department
University of Wales Swansea

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# Introduction: The questions

The topics of this talk are:

- Which categories are **concrete** (more precisely, can be made concrete)?
- Are there *most general* concrete categories (which enable good representations of all concrete categories)?
- Which categories are **algebraic** (more precisely, can be made algebraic)?
- Are there *most general* algebraic categories (which enable good representations of all algebraic categories)?
- Is every concrete category algebraic (i.e., can be made algebraic)?

# Introduction: The answers

We will see:

- There are always non-concretisable categories.
- There are natural most general concrete categories.
- There are natural most general algebraic categories.
- By definition every algebraic category is concrete: Whether every concrete category is algebraic depends on the universe.

# Overview

# Concretisation

- A concrete category over a category $\mathfrak{S}$ is a pair $(\mathfrak{C}, V)$, where $\mathfrak{C}$ is a category and $V : \mathfrak{C} \to \mathfrak{S}$ is a faithful functor.
- A concrete category is a concrete category over $\mathfrak{SET}$.
- A category $\mathfrak{C}$ is **concretisable** if there exists a functor $V$ such that $(\mathfrak{C}, V)$ is a concrete category.
- Examples of concretisable categories are $\mathfrak{CAT}$ and $\mathfrak{SET}^{\mathrm{t}}$.

Which categories are concretisable ?!

# Three sizes of categories

Recall that categories come in three sizes:

1. small categories
2. large categories with small morphism sets
3. large categories with large morphism sets.

The third type is definitely not concretisable. Our aim is to solve the question for the first two types.

# General remarks on concretisation

- $\mathfrak{S}$-Concretisable categories for a concretisable category $\mathfrak{S}$ are again concretisable; we obtain all concretisable categories iff $\mathfrak{SET}$ is $\mathfrak{S}$-concretisable.
- A category is concretisable iff it is *embedabble* into $\mathfrak{SET}$, that is, iff it is isomorphic to a subcategory of $\mathfrak{SET}$. The embedding can be done in such a way that all image sets are non-empty and disjoint.
- Asking for a *full embedding* into $\mathfrak{SET}$ is too strong; for example a category $\mathfrak{C}$ with objects $A, B, C \in \mathrm{Obj}(\mathfrak{C})$ such that

$$|\mathrm{Mor}(B, C)| = 2, \ |\mathrm{Mor}(A, B)| \geq 2$$

has no full embedding into $\mathfrak{SET}$ since for sets $X, Y$ with $|Y^X| = 2$ we have $|X| = 1$ and $|Y| = 2$.

# Small categories are concretisable

Consider a small category $\mathfrak{C}$:

- for $X \in \mathrm{Obj}(\mathfrak{C})$ let

$$V(X) := \{f \in \mathrm{Mor}(\mathfrak{C}) : \mathrm{cod}(f) = X\};$$

- for $f : X \to Y$ in $\mathfrak{C}$ let

$$V(f) := (\eta \in V(X) \mapsto f \circ \eta \in V(Y)).$$

($V$ is not just faithful, but is also an embedding with non-empty disjoint images.)

So concretisation is always possible in the next universe. This leaves open concretisation of large categories with small morphism sets.

# The solution

- Freyd and Isbell provided a necessary and sufficient criterion for a category to be concretisable. (This criterion solves the problem for a fixed category $\mathfrak{SET}_{\mathbb{U}}$ (that is, fixed universe $\mathbb{U}$) and all other categories (possibly from larger universes); however combining categories from different universes seems not really appropriate.)
- In every universe there are non-concretisable categories. (Actually, there seem to be no natural examples?!)

The (pure) concretisation issue seems to be settled.

# Full embeddings

- We we want to represent a category $\mathfrak{C}$ not just by an embedding, but by a *full embedding* into some nice category $\mathfrak{S}$ (that is, $\mathfrak{C}$ shall be isomorphic to a full subcategory of $\mathfrak{S}$).
- The framework for us is given by concretisable categories, i.e., we only want to represent concretisable categories.

A category $\mathfrak{C}$ is called **universal**
iff every concretisable category
has a full embedding into $\mathfrak{C}$.

(Remark: Categories like $\mathfrak{TOP}$ are not universal because of the constant morphisms, while $\mathfrak{MON}$ is not universal because of the null morphisms — recall that discrete categories are concretisable.)

# Isomorphism-closed subcategories



Consider a category $\mathfrak{C}$. Call a sub-category of $\mathfrak{C}$ **isomorphism-closed** (abbreviated by "i-c") if it contains all objects isomorphic (in $\mathfrak{C}$) to some object of that sub-category.

- To specify a full subcategory of $\mathfrak{C}$, it suffices to specify any set of objects.
- Even nicer, to specify an i-c full subcategory it suffices just to specify the isomorphism types.

Accordingly we can consider **i-c full embeddings** instead of just full embeddings; this question seems to have received no attention in the literature, but as a little hobby I will consider it here to some degree.

# Two universal categories

The universal categories $\boldsymbol{S(P^+)}$, $\boldsymbol{S(P^-)}$ are as follows:

1. Objects are pairs $(X, \mathbb{M})$, where $X$ is a set and $\mathbb{M} \in \mathbb{P}(\mathbb{P}(X))$ is a set system on $X$.
2. The morphisms $f : (X, \mathbb{M}) \to (X', \mathbb{M}')$ are maps $f : X \to X'$, "forward morphisms" in $S(P^+)$, i.e.,

$$\forall A \in \mathbb{M} : f(A) \in \mathbb{M}'$$

   and "backward morphisms" in $S(P^-)$, i.e.,

$$\forall A' \in \mathbb{M}' : f^{-1}(A') \in \mathbb{M}.$$

Both categories are obviously concretisable.

- $S(P^+)$ is the category of (arbitrary) hypergraphs with (possibly) infinite hyperedges.
- $S(P^-)$ contains the category of closure systems as an i-c full subcategory (which in turn contains $\mathfrak{TOP}$ as an i-c full subcategory).

# Algebraic categories

Now let's try to make get more specific representing categories.

- An **algebraic category** is a full subcategory of the category of all (single-sorted) algebraic structures of some given type (finitary algebraic operations, arbitrary many).
- To consider multi-sorted algebras does not add anything here.
- Since we allow arbitrary full subcategories, any axiomatically specified category of algebras is included here.

Algebraic categories are concretisable. How special are they?

# Semigroups and monoids

- The category $\mathfrak{SGR}$ of semigroups is algebraic.
- And so is the category $\mathfrak{MON}$ of monoids.

$\mathfrak{MON}$ has a non-full embedding into $\mathfrak{SGR}$ (the forgetful functor). Is there also some full embedding ?!?

There is also a non-full embedding of $\mathfrak{SGR}$ into $\mathfrak{MON}$, given by adjoining a (new) neutral element. Here we see easily that there cannot be a full embedding, since for example

$$\mathrm{Mor}_{\mathfrak{SGR}}((\mathbb{N}_0, +), (\mathbb{N}, +)) = \emptyset.$$

# More example

As for concretisability we have (but this time no so easy to prove):

**Lemma** *Every small category is algebraic.*

So again, going to the next universe we can make a given category algebraic (using a large signature).

Could every concretisable category be algebraic?!

# Algebraic universality

A category is **algebraic-universal** if every algebraic category has a full embedding.

- Are there algebraic-universal categories which are algebraic ?
- And if they exist, do we have universal categories amongst them ?

# A major result

**Theorem** *The category $\mathfrak{SGR}$ of semigroups is algebraic-universal.*

- So there is for example a full embedding of $\mathfrak{MON}$ into $\mathfrak{SGR}$ (it would be nice to see it).
- Using truly large semigroups (in the next universe), thus every category can be represented by semigroups.

What about this universe — is $\mathfrak{SGR}$ also universal ?!?

(It is easy to see that $\mathfrak{SET}$ is fully embedabble into $\mathfrak{SGR}$. But $S(P^{+})$ or $S(P^{-})$ ?!?)

# Filters and ultrafilters

- A **filter** on a set $X$ is a non-empty subset of $\mathbb{P}(X)$ stable under binary intersection and superset-formation; the filter is **proper** if not containing $\emptyset$.
- An **ultrafilter** on $X$ is a filter on $X$ which is maximal amongst all proper filters.
- The set of ultrafilters on $X$ is denoted by $\boldsymbol{\beta X}$.
- For $x \in X$ the generated **trivial** (or **principal**) **ultrafilter** is $\mathfrak{G}\mathfrak{F}(x) := \{A \in \mathbb{P}(X) : x \in A\}$.
- On a finite set $X$ every ultrafilter is trivial, and thus $\mathrm{card}(\beta X) = \mathrm{card}(X)$.
- On an infinite set $X$ we have $\mathrm{card}(\beta X) = 2^{2^{\mathrm{card}(X)}}$.

# More than finite intersections ?

Filters are stable under finite intersections. Can more be said?

1. The elements of $\beta X$ correspond 1-1 to finitely additive $\{0, 1\}$-valued probability measures $\mu$ on $X$.
2. The non-trivial elements of $\beta X$ correspond to those $\mu$ which are not "point measures", i.e., for which $\mu(\{x\}) = 0$ for all $x \in X$.
3. For an infinite cardinal number $\alpha$, the elements of $\beta X$ stable under intersection of at most $\alpha$ elements correspond 1-1 to the $\alpha$-additive $\{0, 1\}$-valued probability measures on $X$.

# Hypothesis (M)

Condition "(M)" on universe $\mathbb{U}$ is:

*In $\mathbb{U}$ there is a cardinal number $\alpha \in \mathbb{U}$, such that in $\mathbb{U}$ every $\alpha$-additive $\{0, 1\}$-valued probability measure is trivial.*

1. For every $\alpha$, the smallest cardinality of a set *X* with a non-trivial $\alpha$-additive $\{0, 1\}$-valued measure is **strongly inaccessible** (that is, is uncountable and is unreachable by union and power-set formation).
2. So if $\mathbb{U}$ does not fulfil (M), then $\mathbb{U}$ contains an unbounded set of strongly inaccessible cardinal numbers.

# A necessary condition for “not (M)”

- The strongly inaccessible cardinal numbers are the cardinalities of universes.
- Call a universe $\mathbb{U}$ a **universe of the second kind** if $\mathbb{U}$ is not just a model of ZFC, but also of ZFCU (that is, besides the stability under all set-theoretic operations, for every $x \in \mathbb{U}$ there exists also a universe $\mathbb{U}'$ with $x \in \mathbb{U}' \in \mathbb{U}$).
- A universe $\mathbb{U}$ is a universe of the second kind iff $\mathbb{U}$ contains an unbounded set of strongly inaccessible cardinal numbers (unbounded within $\mathbb{U}$).

So if $\mathbb{U}$ does not fulfil (M), then $\mathbb{U}$ must be a universe of the second kind.

# About the universality of $\mathfrak{SGR}$

If the universe fulfils (M):

1. *Every concretisable category is algebraic.*
2. So $\mathfrak{SGR}$ is universal.

If the universe does not fulfil (M):

1. $\mathfrak{SET}^{\mathrm{t}}$ is not algebraic.
2. So *no algebraic category is universal.*

End