Stone Duality and the Representation Theorem

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15th December 2008

Boolean Algebras and Topological Spaces

Motivation

We have seen already that **fields of sets** form **boolean algebras**. Another type of set system with similar properties to fields of sets are the **topologies** of **topological spaces**.

Question

Can we form boolean algebras from topological spaces? If so, how?

Well, what do we have in an arbitrary topological space?

- Points in the space.
- Open sets and closed sets.

So, given we already have the notion of a **field of sets** and the open sets form a nice set system, it is natural to consider such a system for the formation of a boolean algebra.

So the topology τ of a topological space (X, τ) is :

- closed under arbitrary union.
- closed under finite intersection.

However, it is *not* in general closed under complementation. So we consider the largest subset of τ which is a field of sets, namely the **clopen sets**.

The B functor

Definition

Given a topological space $Y = (X, \tau)$, we define $B(Y)$ as boolean algebra formed by taking the field of clopen subsets of Y.

Lemma

B has the following properties, given the structure of all topological spaces C and the structure of all boolean algebras D :

1. *For every* $X \in C$, $B(X) \in D$.
2. *For every* $(f : X \to Y) \in C$, $B(f) : B(Y) \to B(X) \in D$
3. $B(\mathrm{id}_X) = \mathrm{id}_{B(X)}$.
4. $B(g \circ f) = B(f) \circ B(g)$

Translations which have such properties are called **contravariant functors**.

Proof. (B is a contravariant functor).

- (1) is trivial (it is clear that $B(X)$ is a field of sets).
- For (2), we have $B(f) = f^{-1}$, so :
  1. $B(f)(\{\}) = f^{-1}(\{\}) = \{\}$ (By Def of inverse image).
  2. $B(f)(A') = f^{-1}(A') = f^{-1}(A)' = B(A)'$ (trivial property of total inverse maps)
  3. $B(f)(A \cup C) = f^{-1}(A \cup C) = f^{-1}(A) \cup f^{-1}(C) = B(f)(A) \cup B(f)(C)$ (trivial property of inverse maps)
  4. $B(f)(A \cap C) = f^{-1}(A \cap C) = f^{-1}(A) \cap f^{-1}(C) = B(f)(A) \cap B(f)(C)$ (trivial property of inverse maps)
- (3) is trivial : $B(\mathrm{id}_X) = \mathrm{id}_X^{-1} = \mathrm{id}_X == \mathrm{id}_{\mathbb{P}(X)}$
- (4) is again trivial : $B(g \circ f) = (g \circ f)^{-1} = f^{-1} \circ g^{-1} = B(f) \circ B(g)$.

□

So we can translate an arbitrary topological space into a boolean algebra, the question now is :

Question

Can we do the reverse? Can we translate a boolean algebra into a topological space?

Problem

Given such a boolean algebra, what can we actually do to form a topological space? Such structures are rather arbitrary.

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Given such a boolean algebra, what can we actually do to form a topological space? Such structures are rather arbitrary.

Solution

Don't look into the individual boolean algebras, but consider the morphisms between them (i.e think from a category theoretical point of view).

The Stone space functor

Definition

Given a boolean algebra A, the Stone space $S(A)$ is the set of homomorphisms from A to B_2 with the induced topology from 2^A.

As a reminder :

Definition

The **cantor space** 2^A *is defined as the set of all functions from* A *to* $B_2 = \{0, 1\}$ *with the topology such that if one fixes an* $a \in A$ *and a* $b \in B_2$ *then the set* $\{f : f(a) = b\}$ *is an open set in the* **subbase** *of* 2^A*, i.e all open sets of* 2^A *may be formed from the finite intersection (to form the* **base***) of elements in the subbase and then arbitrary unions of elements in the* **base***.*

Definition

Given a subset U *of a topological space* (X, τ)*, the* **induced topology** *of* U *from* (X, τ) *is the* $\tau_U := \{U \cap O \mid O \in \tau\}$*, giving the induced topological space* (U, τ_U)*.*

Lemma

S is a contravariant functor from the boolean algebras to the topological spaces.

Proof. (S is a contravariant functor) - Part 1.

- (1) follows from the definition.
- For (2), given a homomorphism f, we have $S(f) : S(B) \to S(A)$ with $S(f)(h) = h \circ f$, and must show $S(f)$ is continuous. It suffices to show that the inverse image of elements in the subbase of $S(A)$ are clopen in $S(B)$. So consider, for some $a \in A$, $b \in B_2$:

$$\begin{aligned} S(f)^{-1}(\{h : h(a) = b\}) &= \{g : S(f)(g) \in h : h(a) = b\} = \\ \{g : S(f)(g)(a) = b\} &= \{g : (g \circ f)(a) = b\} = \\ \{g : g(f(a)) = b\} & \end{aligned}$$

So the inverse image of such a subbase element in $S(A)$ is a subbase element in $S(B)$.

□

Proof. (S is a contravariant functor) - Part 2.

- (3) follows trivially : $S(\mathrm{id}_B)(h) = h \circ \mathrm{id}_B = h = \mathrm{id}_{S(B)}(h)$.
- (4) again follows trivially : $S(g \circ f) = h \circ g \circ f = S(f)(h \circ g) = S(f)(S(g)(h)) = (S(f) \circ S(g))(h)$.

□

Duality and Stone Spaces

The question that now arises is :

Question

Are these functors inverse to each other? Are the boolean algebras and topological spaces dual to each other, i.e is $S(B(A))$ isomorphic to A, and is $S(B(X))$ homeomorphic to X?

Lemma

Given a boolean algebra A, $B(S(A))$ is isomorphic to A.

Proof. ($B(S(A))$ is isomorphic to A).

We define the isomorphism $t_A : A \longrightarrow B(S(A))$ with $t_A(p) = \{f \in S(A) : f(p) = 1\}$:

1. As $\{1\}$ is clopen in B_2, and proj_p is continuous (by definition of 2^X), we have that $\mathrm{proj}_p^{-1}(\{1\}) = \{x \in S(A) : x(p) = 1\}$ is clopen, therefore f is **well-defined** function from A to clopen sets.
2. f is fairly simply shown to be a homomorphism.
3. t_A is injective.
4. t_A is surjective.

□

Lemma

Given a boolean algebra A, for any non-zero $p \in A$, there exists a homomorphism $f : A \longrightarrow B_2$ with $f(p) = 1$.

Proof.

We may apply an extension of Zorn's lemma to get that for every $p \in A$, there is a maximal ideal M of A containing p', and therefore $p \notin M$. Therefore as every maximal ideal is the kernel of some homomorphism into B_2, we have an f with $f[M] = \{0\}$, and therefore $f(p) = 1$. □

Lemma

t_A is **injective***, i.e for any $a, b \in A$ $t_A(a) = t_A(b) \implies a = b$.*

Proof.

From the previous lemma, we have that the kernel of t_A is $\{\emptyset\} = \{0\}$, as for all non-zero p, we know there exists an f that maps it to 0 and so the set $t_A(p)$ is non-empty. Therefore as t_A is a homomorphism, t_A is *injective* (a simple property of boolean homomorphisms). □

Lemma

*t_A is **surjective**, i.e for every clopen set $C \in B(S(A))$, there exists an $a \in A$ such that $t_A(a) = C$.*

Proof.

Consider the clopen sets in $B(S(A))$, as we derive the topology from 2^A, we know that the clopen sets are those that can be formed from finite intersections of the subbase elements, i.e those, that given a fixed $a \in A$ and $b \in B_2$, are of the form $\{f : f(a) = b\}$.
So consider such a subbase element, then we have some a for which all f are fixed to b. If $b = 0$, then $t_A(a') = \{f : f(a) = 0\}$, and if $b = 1$ then $t_A(a) = \{f : f(a) = 1\}$. So we have a mapping for all subbase elements, and as t_A is a homomorphism, finite meets map to finite intersection, and so we may form all clopen sets. □

So in one direction, we have invertibility. B is in some sense the left "pseudo-inverse" of S, i.e we have $S(B(A)) \approx A$.

However, for arbitrary topological spaces, the invertibility doesn't hold :

Counter-example

Consider real line $\mathbb{R}$ with the standard topology. The clopen sets are $\emptyset$ and $\mathbb{R}$ (so we get an algebra isomorphic to B_2) and id is the only homomorphism, so we get the 1-point topology which is clearly not homeomorphic to the standard topology on the reals.

Question

Can we characterise those topological spaces which *are* dual to the boolean spaces?

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Answer

Yes! Consider the properties of 2^X (from which we inherit $S(A)$) :

1. It is compact.
2. It is zero-dimensional (the clopen sets form a base).

Do we get these same properties for $S(A)$?

Lemma

Given an arbitrary boolean algebra A, $S(A)$ is zero-dimensional.

Proof. ($S(A)$ is zero-dimensional).

This follows from the definition, given that 2^X is zero-dimensional, as for any open set O in $S(A)$, $O = H \cap P$ where H is the set of all homomorphisms and P is an open set in 2^A.
Therefore, as P could be formed through arbitrary union of clopen sets $C_i \in 2^A$, then for any such O, we have :

$$O = H \cap P = H \cap (\bigcap_i C_i) = \bigcap_i (H \cap C_i)$$

Where $H \cap C_i$ for each i are clopen sets in $S(A)$. □

Lemma

Given an arbitrary boolean algebra A, $S(A)$ is compact [a].

[a] Note : Here I use the bourbaki notion of compactness - i.e a quasi-compact Hausdorff space

Proof. ($S(A)$ is compact).

One property of compact spaces is that, if we have a closed subset of a compact space, then the topology induced by that subset is compact. Therefore it suffices to show that $S(A)$ is closed in 2^X.

To show that the set of all homomorphisms is closed in the Cantor space, we must simply show that complement, i.e the set of all *non-homomorphisms* is open. A function is not a homomorphism if for some $p, q, p \wedge q$, $f(p \wedge q) \neq f(p) \wedge f(q)$, and so we may fix $f(p \wedge q), f(p)$ and $f(q)$ in just such a way. Fixing those finite number of value yields an open set, and the union of all such sets yields all *non-homomorphisms*. □

Question

Is this all that is necessary? Are all zero-dimensional compact topological spaces homeomorphic to the dual of some boolean algebra?

Answer

Yes!

Lemma

Given a topological space X, $S(B(X))$ is homeomorphic to X.

Proof.

We define the homeomorphism $i_X : X \to S(B(X))$ with $i_X(p)(C) = p \in C$ for some clopen set $C \in B(X)$:

- i_X is continuous, as by the previous proof, we have that all clopen sets in $S(B(X))$ are of the form $\{f : f(P) = 1\}$ for some $P \in B(X)$, and as $B(X)$ contains all clopen sets of X, we have that the inverse image of the clopen sets are clopen.
- i_X is surjective.
- i_X is injective.

□

Lemma

i_X is surjective.

Proof. (i_X is surjective) - Part 1.

Again, as we know all clopen sets in $S(B(X))$ are of the form $\{f : f(P) = 1\}$, we have that for such a non-empty set, there is a non-empty clopen set $P \subseteq X$. Therefore, as the clopen sets form a base of $S(B(X))$, every open set must contain some point in $\mathrm{Rg}(i_X)$ (i.e $\mathrm{Rg}(X)$ is **dense** in $S(B(X))$).

The fact that $\mathrm{Rg}(i_X)$ is dense in $S(B(X))$ means that for any non-empty open set $O \subseteq S(B(X))$, we have that $i_X^{-1}(O)$ is non-empty. □

Proof. (i_X is surjective) - Part 2.

Consider all clopen neighbourhoods C_i of $h \in S(B(X))$:

- The inverse image of any finite intersection of these C_i is non-empty (as any open set in $S(B(X))$ is non-empty).
- By the dual notion of compactness for closed sets, we have that, since $i_X^{-1}(C_i)$ is closed for any C_i, and all finite intersections of $i_X^{-1}(C_i)$ are non-empty, then the intersection of for all i of $i_X^{-1}(C_i)$ is non-empty.
- Therefore, as by the Hausdorff property and zero-dimensionality of $S(B(X))$, we have that for every such h, we can find a clopen neighbourhood of h that separates it from any other point h_2, we have that, the intersection of all such C_i is $\{h\}$, and therefore, the above states that $i_X^{-1}(h)$ is non-empty, and therefore i_X is surjective.

Lemma

i_X is injective.

Proof.

For injectivity, we need $i_X(p_1) = i_X(p_2) \implies p_1 = p_2$, so we need the existence of a $C \in B(X)$ such that :

$$i_x(p_1)(C) \neq i_X(p_2)(C)$$

So that $p_1 \in C$ but $p_2 \notin C$. As $S(B(X))$ is zero-dimensional Hausdorff space, we can find two disjoint neighbourhoods $p \in N_1$ and $p_2 \in N_2$(by Hausdorff), and as these neighbourhoods are formed from clopen sets (by zero-dimensionality), then by compactness (so we gain closure of both open and closed sets under finite union and intersection) there exists a clopen set $p \in C \subseteq N_1$, then we have $p_2 \notin C$. □

Naturality

One additional property that one would like is that the the isomorphisms that have been defined are not in some way special, i.e that they interact naturally with other morphisms. So given :

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ t_A \downarrow & & \downarrow t_B \\ B(S(A)) & \xrightarrow{B(S(f))} & B(S(C)) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \downarrow & & \downarrow i_Y \\ S(B(X)) & \xrightarrow{S(B(f))} & S(B(Y)) \end{array}$$

where A and B are boolean algebras, we want :
$(t_B \circ f)(a) = (B(S(f)) \circ t_A)(a)$ and
$(i_Y \circ f)(x)(C) = (S(B(f)) \circ i_X)(x)(C)$. These follow trivially in this case.

The Stone Duality

The proceeding result, i.e that for every boolean algebra there is a dual space and for every topological space there is a dual algebra is known as the **Stone duality**.

Representation theorems and examples

Immediately from the proof that t_A is an isomorphism we get the **Representation Theorem for boolean algebras** :

Corollary (Representation Theorem for boolean algebras)

Every boolean algebra is isomorphic to a field of sets.

Based on the proof that i_A is a homeomorphism, we get a similar **representation theorem** for the Stone spaces :

Corollary (Representation theorem for Stone spaces)

Every stone space is homeomorphic to a subspace of some Cantor space.

Question

Given that we now know that such a duality exists, can we dualise certain notions from one structure to the other?

Can we take some notion in the boolean algebras and find a similar notion in the boolean spaces, or vice versa?

Does this duality help us understand one structure through dualisation into the other?

A simple example in this case is that of the finite boolean algebras :

Lemma

The duals of finite boolean algebras are exactly the finite discrete spaces.

Proof.

- Given a finite algebra A, there are only finitely many homomorphisms to B_2, and given the induced topology from 2^X, we have sets containing all homomorphisms which fix a finite subset of A to values in B, but as A is finite, this gives us all subsets, and so $S(A)$ is discrete.
- Given a finite discrete space X, the powerset $\mathbb{P}(X)$ is finite, and contains only clopen sets, therefore $B(X)$ is a finite boolean algebra.

□

Non-trivial dual notions

A more interesting question is to consider one of the more important structures in the boolean algebras, that of the ideals.

Question

What notion within the topological spaces, dualises the notion of an ideal?

Given a boolean algebra A with ideal I, consider the following :

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A/I \\ \downarrow S & & \downarrow S \\ S(A) & \xleftarrow{S(\pi)} & S(A/I) \end{array}$$

Consider the range of $S(\pi)$ in $S(A)$. Given that every compact subspace of a Hausdorff space is closed, and $S(\pi)$ forms a subspace in $S(A)$, we have for every ideal I an associated closed set.
As every closed set has as it's complement an open set, one may also associate with every ideal, an open set.

As a final example :

Question

Given the powerset fields of sets, i.e $\mathbb{P}(X)$ for some set X, what topological spaces do these represent?

Ultrafilters

Consider the definition, the elements of $Y \in \mathbb{P}(X)$ are mapped to homomorphisms in $S(\mathbb{P}(X))$ of the form $h : A \to B_2 \mid h(Y) = 1$.

One may remember, the correspondence between maximal ideals and the kernels of homomorphisms into B_2. The dual of this is simply that maximal filters, so called **ultrafilters** have the same property, i.e they are the sets of elements that map to 1 in some homomorphism to B_2.

In particular, if one considers all the homomorphisms in $S(\mathbb{P}(Y))$, one can easily see (from the nature of $\mathbb{P}(Y)$) that one gets a representation (as a homomorphism) of all ultrafilters of Y.

Stone-Čech compactification

This is in particular useful, as then for an arbitrary discrete space X, as $S(B(X))$ must be compact, one may take $S(B(X))$ to get a compactification of X.

Such a compactification formed by taking all ultrafilters of a discrete space is called the **Stone-Čech compactification**.

End.