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Department of Computer Science

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CS_376Programming with Abstract Data Types

Solutions to Coursework 1

Question 1.

(a) $\mathsf{reg}(s,c) := \forall l_1, l_2\, (\neg\mathsf{att}(s,l_1,c) \land \neg\mathsf{att}(s,l_2,c) \to l_1 = l_2)$.

Alternative solution: $\mathsf{reg}'(s,c) := \forall l_1, l_2\, (\neg l_1 = l_2 \to \mathsf{att}(s,l_1,c) \lor \mathsf{att}(s,l_2,c))$.

Additional exercise: prove that both solutions are logically equivalent.

(b) $\mathsf{model}(s) := \forall c_1, c_2, c_3, l_1, l_2, l_3\, (\neg\mathsf{att}(s,l_1,c_1) \land \neg\mathsf{att}(s,l_2,c_2) \land \neg\mathsf{att}(s,l_3,c_3) \to$
$(l_1 = l_2 \land c_1 = c_2) \lor (l_1 = l_3 \land c_1 = c_3) \lor (l_3 = l_2 \land c_3 = c_2))$.

(c) $\mathsf{lazy}(s) := \neg\exists c\, \mathsf{reg}(s,c)$.

(d) $\neg\exists s\, \forall c\, \mathsf{reg}(s,c)$.

(e) $\forall c\, (\exists s\, \mathsf{reg}(s,c) \to \exists l\, \forall s\, (\mathsf{att}(s,l,c)))$.

Question 2.

(a) This holds in Y2, but is not logically valid. Here is a description of a situation (= algebra) where the formula is false: Suppose there is only one course, and there is a student, say John, who missed exactly two lectures of that course. Then John is a model student (he missed only 2 lectures in total), but he is lazy (he attended no course regularly).

(b) This is logically valid (and hence holds in Y2). Informal proof: If student s attends cs221 regularly, then it is obviously not true that she attended no course regularly, hence she is not lazy.

Here is a proof in minimal logic. Recall that $\mathsf{lazy}(s) = \neg\exists c\, \mathsf{reg}(s,c) = \exists c\, \mathsf{reg}(s,c) \to \bot$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{v : \mathsf{lazy}(s) \qquad \dfrac{u : \mathsf{reg}(s,\mathsf{cs221})}{\exists c\, \mathsf{reg}(s,c)}\,\exists^+}{\bot}\,\to^-}{\neg\mathsf{lazy}(s)}\,\to^+ v : \mathsf{lazy}(s)}{\mathsf{reg}(s,\mathsf{cs221}) \to \neg\mathsf{lazy}(s)}\,\to^+ u : \mathsf{reg}(s,\mathsf{cs221})}{\forall s\, (\mathsf{reg}(s,\mathsf{cs221}) \to \neg\mathsf{lazy}(s))}\,\forall^+
$$

Question 3.

(a)

$$\dfrac{\dfrac{\dfrac{u : P \land Q}{P}\,\land^-_{\mathrm{l}}}{P \lor Q}\,\lor^+_{\mathrm{l}}}{P \land Q \to P \lor Q}\,\to^+ u : P \land Q$$

(b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u : P \leftrightarrow \neg P}{P \to \neg P}\,\land^-_{\mathrm{l}} \quad v : P}{\neg P}\,\to^- \quad v : P}{\bot}\,\to^-}{\neg P}\,\to^+ v : P \qquad \dfrac{\dfrac{u : P \leftrightarrow \neg P}{\neg P \to P}\,\land^-_{\mathrm{r}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \leftrightarrow \neg P}{P \to \neg P}\,\land^-_{\mathrm{l}} \quad w : P}{\neg P}\,\to^- \quad w : P}{\bot}\,\to^-}{\neg P}\,\to^+}{P}\,\to^-}{\bot}\,\to^-}{\neg(P \leftrightarrow \neg P)}\,\to^+ u : P \leftrightarrow \neg P$$

(c)

$$\dfrac{\dfrac{\dfrac{u : P \lor Q \qquad \dfrac{\dfrac{\dfrac{v : \neg P \quad w : P}{\bot}\,\to^-}{Q}\,\mathrm{efq}}{P \to Q}\,\to^+ w : P \qquad \dfrac{u_1 : Q}{Q \to Q}\,\to^+ u_1 : Q}{Q}\,\lor^-}{\neg P \to Q}\,\to^+ v : \neg P}{(P \lor Q) \to (\neg P \to Q)}\,\to^+ u : P \lor Q$$

(d)

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\forall x\, P(x) \qquad \dfrac{\dfrac{\dfrac{\dfrac{v : \neg\exists x\, \neg P(x) \quad \dfrac{w : \neg P(x)}{\exists x\, \neg P(x)}\,\exists^+}{\bot}\,\to^-}{\neg\neg P(x)}\,\to^+ w : \neg P(x)}{P(x)}\,\mathrm{raa}}{\forall x\, P(x)}\,\forall^+}{\bot}\,\to^-}{\neg\neg\exists x\, \neg P(x)}\,\to^+ v : \neg\exists x\, \neg P(x)}{\exists x\, \neg P(x)}\,\mathrm{raa}}{\neg\forall x\, P(x) \to \exists x\, \neg P(x)}\,\to^+ u : \neg\forall x\, P(x)$$

Question 4.

Induction on P.

Case P is $\bot$:

$$\dfrac{\dfrac{u : \neg\neg\bot \qquad \dfrac{v : \bot}{\neg\bot}\;\to^+ v : \bot}{\bot}\;\to^-}{\neg\neg\bot \to \bot}\;\to^+ u : \neg\neg\bot$$

Case P is an equation, $s = t$:

$$\dfrac{\dfrac{\mathsf{Stability} : \forall x, y\,(\neg\neg\, x = y \to x = y)}{\forall y\,(\neg\neg\, s = y \to s = y)}\;\forall^-}{\neg\neg s = t \to s = t}\;\forall^-$$

Case P is of the form $Q \wedge R$:

$$\dfrac{\dfrac{\dfrac{\neg\neg Q \to Q \text{ (i.h.)} \qquad \dfrac{\dfrac{u : \neg\neg(Q \wedge R) \qquad \dfrac{\dfrac{v : \neg Q \qquad \dfrac{w : Q \wedge R}{Q}\;\wedge^-_1}{\bot}\;\to^-}{\neg(Q \wedge R)}\;\to^+ w : Q \wedge R}{\bot}\;\to^-}{\neg\neg Q}\;\to^+ v : \neg Q}{Q}\;\to^- \qquad R\ (\textbf{similarly})}{Q \wedge R}\;\wedge^+}{\neg\neg(Q \wedge R) \to Q \wedge R}\;\to^+ u : \neg\neg(Q \wedge R)$$

Case P is of the form $Q \to R$:

$$
\begin{prooftree}
\AxiomC{$\neg\neg R \to R$ (i.h.)}
\AxiomC{$u : \neg\neg(Q \to R)$}
\AxiomC{$v : \neg R$}
\AxiomC{$w : Q \to R$}
\AxiomC{$u1 : Q$}
\RightLabel{$\to^-$}
\BinaryInfC{R}
\RightLabel{$\to^-$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ w : Q \to R$}
\UnaryInfC{$\neg(Q \to R)$}
\RightLabel{$\to^-$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ v : \neg R$}
\UnaryInfC{$\neg\neg R$}
\RightLabel{$\to^-$}
\BinaryInfC{R}
\RightLabel{$\to^+ u1 : Q$}
\UnaryInfC{$Q \to R$}
\RightLabel{$\to^+ u : \neg\neg(Q \to R)$}
\UnaryInfC{$\neg\neg(Q \to R) \to Q \to R$}
\end{prooftree}
$$

Case P is of the form $\forall x\, Q(x)$:

$$
\begin{prooftree}
\AxiomC{$\neg\neg Q(x) \to Q(x)$ (i.h.)}
\AxiomC{$u : \neg\neg\forall x\, Q(x)$}
\AxiomC{$v : \neg Q(x)$}
\AxiomC{$w : \forall x\, Q(x)$}
\RightLabel{$\forall^-$}
\UnaryInfC{$Q(x)$}
\RightLabel{$\to^-$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ w : \forall x\, Q(x)$}
\UnaryInfC{$\neg\forall x\, Q(x)$}
\RightLabel{$\to^-$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ v : \neg Q(x)$}
\UnaryInfC{$\neg\neg Q(x)$}
\RightLabel{$\to^-$}
\BinaryInfC{$Q(x)$}
\RightLabel{$\forall^+$}
\UnaryInfC{$\forall x\, Q(x)$}
\RightLabel{$\to^+ u : \neg\neg\forall x\, Q(x)$}
\UnaryInfC{$\neg\neg\forall x\, Q(x) \to \forall x\, Q(x)$}
\end{prooftree}
$$