

CS_376 Programming with Abstract Data Types

Coursework 2

Question 1. Let Σ be the signature consisting of one sort s , a constant $c: s$ and one binary operation $f: s \times s \rightarrow s$. Consider the following Σ -algebras A, B, C :

$A_s := \mathbf{R}$ = the set of real numbers,
 $c^A := 0$,
 $f^A(r_1, r_2) := r_1 + r_2$ ($r_1, r_2 \in \mathbf{R}$), that is, f^A is addition.

$B_s := \mathbf{R}_0^+ = \{r \in \mathbf{R} \mid r \geq 0\}$,
 $c^B := 0$,
 $f^B(r_1, r_2) := r_1 + r_2$ ($r_1, r_2 \in \mathbf{R}_0^+$).

$C_s := \mathbf{R}^+ = \{r \in \mathbf{R} \mid r > 0\}$,
 $c^C := 1$,
 $f^C(r_1, r_2) := r_1 * r_2$ ($r_1, r_2 \in \mathbf{R}^+$), that is, f^C is multiplication.

For each of the algebras B and C decide whether or not it is isomorphic to the algebra A . Justify your answers. [20 marks]

Question 3.

- (a) Produce an initial specification for the algebra of *nonempty finite sets of natural numbers* that has the following operations:
- **singleton**: creating a set with one element;
 - **union**: the union of two sets;
 - **element**: testing whether a number is an element of a set.
 - **card**: the cardinality of a set;

You may import suitable data types of natural numbers and booleans, and add further operations. You may also use familiar mathematical notation. For example, you may write $\{x\}$ for `singleton(x)`, $s \cup t$ for `union(s, t)`, e.t.c. [30 marks]

- (b) Determine a minimal set of generators in the specification you have given in (a) above. Are these generators free? [5 marks]
- (c) Suppose you added an operation $\text{count} : \text{set} \rightarrow \text{nat}$ and the following equations to your initial specification:

$$\begin{aligned}\text{count}(\{x\}) &= 1 \\ \text{count}(s \cup t) &= \text{count}(s) + \text{count}(t)\end{aligned}$$

Explain why the extended specification would be flawed. [5 marks]

Question 4. Let s be a sort, $1 : s$ a constant, $*$, $\text{exp} : \text{nat} \times s \rightarrow s$ operations ($*$ used in infix notation), and $x, y, z : s$ variables. Consider the term rewriting system R given by the following rules:

$$x * 1 \mapsto x$$

$$1 * x \mapsto x$$

$$\text{exp}(1, x) \mapsto 1$$

$$\text{exp}(x, 1) \mapsto x$$

$$\text{exp}(x * y, z) \mapsto \text{exp}(x, z) * \text{exp}(y, z)$$

$$\text{exp}(x, y * z) \mapsto \text{exp}(\text{exp}(x, y), z)$$

Prove that R is terminating. [40 marks]

Date due: Tuesday, 8 December 2008