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December 10, 2009

CS_376 Programming with Abstract Data Types
Solutions to Coursework 2

Question 1. The algebras A and B are not isomorphic since they do not satisfy the same Σ-formulas. For example, $\forall x\, \exists y\, f(x,y) = c$ holds in A, but not in B.

The algebras A and C are isomorphic. Here is an isomorphism:

$$\varphi : \mathbf{R} \to \mathbf{R}^+, \quad \varphi(x) := e^x$$

We know from school mathematics:

1. $\varphi(x) > 0$ for all $x \in \mathbf{R}$ (hence $\varphi\colon \mathbf{R} \to \mathbf{R}^+$ is well-defined) and φ is bijective (the natural logarithm is its inverse).
2. $\varphi(0) = 1$ and $\varphi(x + y) = \varphi(x) * \varphi(y)$, hence φ is a homomorphism from A to C.

Question 2. (a)

Init Spec	NONEMPTYSET
Sorts	$\mathsf{boole}, \mathsf{nat}, \mathsf{set}$
Constants	$0\colon \mathsf{nat},\ \mathsf{T}, \mathsf{F}\colon \mathsf{boole}$
Operations	$\mathsf{succ}\colon \mathsf{nat} \to \mathsf{nat}$ $\mathsf{eq}\colon \mathsf{nat} \times \mathsf{nat} \to \mathsf{boole}$ $\mathsf{or}\colon \mathsf{boole} \times \mathsf{boole} \to \mathsf{boole}$ $\mathsf{if}\colon \mathsf{boole} \times \mathsf{set} \times \mathsf{set} \to \mathsf{set}$ $\mathsf{singleton}\colon \mathsf{nat} \to \mathsf{set}$ $\mathsf{union}\colon \mathsf{set} \times \mathsf{set} \to \mathsf{set}$ $\mathsf{member}\colon \mathsf{nat} \times \mathsf{set} \to \mathsf{boole}$ $\mathsf{card}\colon \mathsf{set} \to \mathsf{nat}$
Variables	$x, y\colon \mathsf{nat}, \quad b\colon \mathsf{boole}, \quad s, t, u\colon \mathsf{set}$
Equations	$\mathsf{eq}(x,x) = \mathsf{T}$ $\mathsf{eq}(0, \mathsf{succ}(x)) = \mathsf{F}$ $\mathsf{eq}(\mathsf{succ}(x), 0) = \mathsf{F}$ $\mathsf{eq}(\mathsf{succ}(x), \mathsf{succ}(y)) = \mathsf{eq}(x,y)$ $\mathsf{or}(\mathsf{T}, b) = \mathsf{T}$ $\mathsf{or}(b, \mathsf{T}) = \mathsf{T}$ $\mathsf{or}(\mathsf{F}, \mathsf{F}) = \mathsf{F}$ $\mathsf{if}(\mathsf{T}, s, t) = s$ $\mathsf{if}(\mathsf{F}, s, t) = t$ $\mathsf{union}(s, s) = s$ $\mathsf{union}(s, t) = \mathsf{union}(t, s)$ $\mathsf{union}(\mathsf{union}(s,t), u) = \mathsf{union}(s, \mathsf{union}(t, u))$ $\mathsf{member}(x, \mathsf{singleton}(y)) = \mathsf{eq}(x, y)$ $\mathsf{member}(x, \mathsf{union}(s,t)) = \mathsf{or}(\mathsf{member}(x,s), \mathsf{member}(x,t))$ $\mathsf{card}(\mathsf{singleton}(x)) = \mathsf{succ}(0)$ $\mathsf{card}(\mathsf{union}(\mathsf{singleton}(x), s)) = \mathsf{if}(\mathsf{member}(x,s), \mathsf{card}(s), \mathsf{succ}(\mathsf{card}(s)))$

(b) A minimal set of generators is $\{0, \mathsf{succ}, \mathsf{T}, \mathsf{F}, \mathsf{singleton}, \mathsf{union}\}$. The generators are not free, since different generator terms may denote the same set. For example, the generator terms $\mathsf{union}(\mathsf{singleton}(0), \mathsf{singleton}(0))$ and $\mathsf{singleton}(0)$ are different, but denote the same set $\{0\}$.

(c) From the equations for count and the equations in (a) it follows (assuming suitable equations for addition)

$$\mathsf{succ}(0) = \mathsf{count}(\{0\}) = \mathsf{count}(\{0\} \cup \{0\}) = \mathsf{count}(\{0\}) + \mathsf{count}(\{0\}) = \mathsf{succ}(0) + \mathsf{succ}(0) = \mathsf{succ}(\mathsf{succ}(0))$$

which is a new equation for natural numbers. In fact, one could prove that all natural numbers are equal (similarly for booleans and sets). Hence the model of the extended initial specification collapses.

Question 3. The given term rewriting system:

$$x * 1 \mapsto x$$

$$1 * x \mapsto x$$

$$\mathsf{exp}(1, x) \mapsto 1$$

$$\mathsf{exp}(x, 1) \mapsto x$$

$$\mathsf{exp}(x * y, z) \mapsto \mathsf{exp}(x, z) * \mathsf{exp}(y, z)$$

$$\mathsf{exp}(x, y * z) \mapsto \mathsf{exp}(\mathsf{exp}(x, y), z)$$

To prove termination we use an interpretation in the following algebra A:

$$A_s := \{2, 3, 4, \ldots\}, \quad 1^A := 2, \quad n *^A m := n * m + 1, \quad \mathsf{exp}^A(n, m) := n^m.$$

Clearly, the operations are monotone. It remains to be checked that $l^{A,\alpha} > r^{A,\alpha}$ for every rule $l \mapsto r$ and every assignment $\alpha\colon \mathsf{Variables} \to A$. Hence, one has to verify for all $m, n, k \in \{2, 3, 4, \ldots\}$ the following inequalities:

$$\begin{array}{lcl}
n *^A 1^A > n & & n * 2 + 1 > n \\
1^A *^A n > n & & 2 * n + 1 > n \\
\mathsf{exp}^A(1^A, n) > 1^A & \text{that is} & 2^n > 2 \\
\mathsf{exp}^A(n, 1^A) > n & & n^2 > n \\
\mathsf{exp}^A(n *^A m, k) > \mathsf{exp}^A(n, k) *^A \mathsf{exp}^A(m, k) & & (n * m + 1)^k > n^k * m^k + 1 \\
\mathsf{exp}^A(n, m *^A k) > \mathsf{exp}^A(\mathsf{exp}^A(n, m), k) & & n^{m*k+1} > (n^m)^k
\end{array}$$

Clearly, all inequalities hold.