Dynamic Definability

Sebastian Siebertz
TU Berlin

Erich Grädel
RWTH Aachen

We investigate the logical resources required to maintain knowledge about a property of a structure under changing operations after a logical initialization. We are especially interested in properties which are not expressible in first-order logic or first-order logic plus counting, i.e. in properties which cannot be queried by relational calculus or SQL, respectively, in a relational database. An example for such a query is the transitive closure of a graph. In a setting where the database changes frequently, it may be beneficial to initialize the answer to the query once and maintain additional information so that after each change of the database the query can be ansered efficiently without recomputing it from scratch. The auxiliary information needs to be updated after every change of the database in order to be consistent with the database. Surprisingly, many problems which cannot be queried with FO or FO+C can indeed be handled with FO- or FO+C- updates in the dynamic context.

Dynamic algorithms (i.e. algorithms maintaining dynamic data structures) for concrete problems have been studied for over three decades and many sophisticated techniques have been developed. A general theory of dynamic complexity providing structural results was first introduced by Miltersen et al. and Patnaik and Immerman. Motivated by the success of the descriptive approach to traditional complexity theory, Patnaik and Immerman provided a descriptive approach to dynamic complexity theory. Their setting, called *Dyn-FO*, is designed to handle classes of structures, and auxiliary data consists of relations over the universe of the structures. In particular, no high level datastructures are used and only a polynomial amount of auxiliary space is available. Weber and Schwentick provide an elegant generalization to this setting which clarifies the role of precomputation. The initialization of the auxiliary structure and all updates are handled by logical interpretations. This allows for an analysis with well-known methods from finite model theory, and furthermore, all results obtained are readily usable for relational databases. A similar formalism restricted to database applications, called *FOIES*, is defined and studied by Dong, Su, Libkin and Wong.

We adapt the descriptive setting of Weber and Schwentick, i.e. we consider finite relational structures where auxiliary data consists of relations over the domain of the input structure. Initialization of the auxiliary structure and all updates are handled by logical interpretations. In contrast to the investigations of Patnaik and Immerman and Weber and Schwentick, the emphasis is on logical definability in the setting of not necessarily ordered structures. In this setting definability and computational complexity are not strictly linked by the well-known capturing results of descriptive complexity theory.

We show that many upper bounds for concrete problems can be translated to our setting. Some dynamic systems from the literature directly translate into our setting, e.g. the incremental evaluation system for reachability in acyclic directed graphs by Dong and Su, whereas others rely crucially on the availability of an ordering, e.g. the dynamic algorithm for reachability in undirected graphs by Patnaik and Immerman. Due to its practical relevance we concentrate in particular on the reachbility problem. We show that the undirected reachability query can be handled with first-order updates in the unordered setting. We further establish a suitable reduction concept which is slightly more complicated than in the ordered setting and show that many problems reduce to undirected reachability via dynamic reductions. These include bipartiteness, k-vertex disjoint paths, k-edge connectivity and clique cover-2.

We even show that the auxiliary relations for all of the above problems can be maintained independent of the update order. A dynamic system with this property is called deterministic by Dong and Su and memoryless by Patnaik and Immerman. The advantage of such deterministic systems in the unordered setting is that, in contrast to nondeterministic systems, they can deal with arbitrary fixed numbers of updates at the same time and thus allow for a simpler reduction concept. This answers an open question by Dong and Su. We further show how to modify the algorithm of Etessami to handle tree isomorphism with $\mathsf{FO{+}C}$-updates in a deterministic fashion.

Providing lower bounds on the power of dynamic systems on ordered structures amounts to providing lower bounds for a general model of computation. It was shown by Etessami that this is also the case when the dynamic process starts with an empty structure, as order and arithmetic can be built incrementally. Thus proving lower bounds in the ordered setting is probably beyond reach. The situation is different when the dynamic process starts with a non-empty unordered structure. We show that in our unordered setting neither equal cardinality nor tree isomorphism can be handled with FO-updates. Our method is similar to that of Dong and Su, using Ehrenfeucht-Fraïssé games. This separates dynamic systems with FO-updates from those with $\mathsf{FO{+}C}$-updates. The separation on ordered structures, i.e. the separation of systems with AC^0-updates from systems with TC^0-updates, is a major open problem in dynamic complexity theory. Our result also implies that there is no deterministic *FOIES* in the notion of Dong and Su for equal cardinality and tree isomorphism. The former was already known, the latter is the first result of nonexistence of a deterministic *FOIES* for a query over non-unary input vocabulary.