

Coinduction and program extraction in computable analysis

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Outline

Introduction

A coinductive description of approximable real numbers

Program extraction in computable analysis

Conclusion

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A coinductive description of approximable real numbers

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The Curry-Howard correspondence (or Brouwer-Heyting-Kolmogorov interpretation)

Formulas correspond to data types

Proofs correspond to programs

$A \vee B$	disjoint sum
$A \wedge B$	cartesian product
$A \rightarrow B$	function space
$\exists x A$	(dependent) cartesian product
$\forall x A$	(dependent) function space

A proof of a formula A corresponds to a program constructing an element of A .

- ▶ What is a function?
- ▶ What if the quantified x ranges over abstract objects?
- ▶ How do we interpret logical axioms, e.g. $A \vee \neg A$?
- ▶ How do we interpret maths axioms, e.g. induction, choice?
- ▶ Why is it interesting and useful?

Why Curry-Howard is interesting and useful

Foundations

Constructive foundation of Mathematics (Brouwer, Heyting, Kolmogorov, Gödel, Kleene, Kreisel, Martin-Löf). Properties of logical and mathematical systems (Realizability \Rightarrow existence and disjunction property; Dialectica Interpretation \Rightarrow consistency)

Programming

Program extraction (Minlog, Coq, Isabelle, Agda). In Minlog, realizability is used to automatically extract from a proof a program and its correctness proof.

Mathematics

Approximation-, fixedpoint-, ergodic-theory (Kohlenbach, Avigad, . . . , using DI). The study of function spaces led to new developments in computability theory, topology, domain theory. The problem of C-H interpreting classical choice axioms has led to new recursion principles such as bar recursion and products of selection functions (see recent work by Martin Escardo and Paulo Oliva).

What is a function and when is it a proof of an implication?

BHK-interpretation: A proof of $A \rightarrow B$ is a function f mapping proofs of A to proofs of B .

- ▶ f should be computable. What does this mean if A itself consists of functions? (\Rightarrow computability in higher types)
- ▶ Don't we need a *proof* that f does it's job? (circularity!)

Realizing an implication

Realizability (Kleene, Kreisel)

$$f \mathbf{r} (A \rightarrow B) \equiv \forall a (a \mathbf{r} A \rightarrow f(a) \mathbf{r} B)$$

Dialectica Interpretation (Gödel)

Define realizability such that $a \mathbf{r} A$ always is a purely universal formula $\forall u (a \mathbf{r}_u A)$ with quantifier free kernel $a \mathbf{r}_u A$.

$$\begin{aligned} f \mathbf{r} (A \rightarrow B) &\equiv \forall a (\forall u (a \mathbf{r}_u A) \rightarrow \forall v (b \mathbf{r}_v B)) \\ &\equiv \forall a \forall v (\forall u (a \mathbf{r}_u A) \rightarrow b \mathbf{r}_v B) \end{aligned}$$

By a continuity argument the premise, $\forall u (a \mathbf{r}_u A)$, is needed for finitely many u only. In fact, classically, a single u , to be computed from a and v , suffices:

$$(f, g) \mathbf{r} (A \rightarrow B) \equiv \forall a, v (a \mathbf{r}_{g(a,v)} A \rightarrow f(a) \mathbf{r}_v B)$$

Soundness

Both interpretations, Realizability and the Dialectica Interpretation, extract from a proof of A a term M and a proof of $M \Vdash A$ (Soundness Theorem).

In the Dialectica Interpretation the proof of $M \Vdash A$ takes place in a quantifier free system!

In the following we will work with realizability.

Realizing quantifiers

Traditionally:

$$\begin{aligned}(x, a) \mathbf{r} \exists x A(x) &\equiv a \mathbf{r} A(x) \\ f \mathbf{r} \forall x A(x) &\equiv \forall x (f(x) \mathbf{r} A(x))\end{aligned}$$

x may range over abstract object (reals, real functions, ...).

This seems to require a realizing programming language with data types for such abstract objects.

Alternative: uniform realization of quantifiers

$$\begin{aligned}a \mathbf{r} \exists x A(x) &\equiv \exists x (a \mathbf{r} A(x)) \\ a \mathbf{r} \forall x A(x) &\equiv \forall x (a \mathbf{r} A(x))\end{aligned}$$

Relativized quantifiers

For concrete objects we may relativize the quantifiers.

For example, “every natural number can be approximately halved” can be expressed by

$$\forall x (\mathbb{N}(x) \rightarrow \exists y (\mathbb{N}(y) \wedge (x = 2y \vee x = 2y + 1)))$$

where the predicate \mathbb{N} is defined in such a way that $n \mathbf{r} \mathbb{N}(x)$ means that n is a representation of the natural number x .

From a (constructive) proof of this formula, realizability extracts a program of type $\mathbf{N} \rightarrow \mathbf{nat} \times \mathbf{B}$ computing integer division by 2.

Program extraction and the law of excluded middle (LEM)

Realizing, say, $\forall x (\mathbb{N}(x) \rightarrow A(x) \vee \neg A(x))$ would mean to construct a program computing for every (representation of) a natural number x a realizer of $A(x)$ or a realizer of $\neg A(x)$. This is impossible, in general.

But, one can eliminate all uses of LEM in proofs of formulas of the form

$$\forall x (\mathbb{N}(x) \rightarrow \exists y (\mathbb{N}(y) \wedge A_0(x, y)))$$

where $A_0(x, y)$ is decidable, using Gödel's negative translation and the Friedman/Dragalin A -translation.

Other approaches to program extraction from classical proofs

- ▶ ϵ -substitution calculus (Hilbert).
- ▶ Interpretation of $\neg\neg A \rightarrow A$ by continuations (Felleisen).
- ▶ Direct computational interpretation of classical sequent calculus ($\lambda\mu$ -calculus, Parigot).
- ▶ Interpretation of restricted forms of LEM by learning based realizability (Berardi, Aschieri)
- ▶ Realizability interpretation of classical systems via stacks and processes (Krivine).

Interpreting induction

Induction on natural numbers

$$A(0) \wedge \forall x (A(x) \rightarrow A(x + 1)) \rightarrow \forall x (\mathbb{N}(x) \rightarrow A(x))$$

is a special case of induction on an inductively defined predicate:

Set $\Phi(X) := \{0\} \cup \{x + 1 \mid x \in X\}$, then $\mathbb{N} = \mu\Phi = \mu X.\Phi(X)$

In general, one has for a monotone predicate transformer Φ an induction schema for its least fixed point $\mu\Phi$:

$$\Phi(\mathcal{P}) \subseteq \mathcal{P} \rightarrow \mu\Phi \subseteq \mathcal{P}$$

The data type associated with $\mu\Phi$ is the initial algebra

$\text{In}_\varphi : \varphi(\mu\varphi) \rightarrow \mu\varphi$ of a functor φ derived from Φ . The induction scheme is realized by the iterator It_φ that iterates any “step

function” (i.e. φ -algebra) $f : \varphi(\alpha) \rightarrow \alpha$ to an algebra morphism

$\text{It}_\varphi(f) : \mu\varphi \rightarrow \alpha$ with computation rule (i.e. morphism equation)

$$\text{It}_\varphi(f) \text{In}_\varphi(m) = f(\mathbf{map}_\varphi(\text{It}_\varphi(f))(m))$$

Example: Natural numbers

Recall $\mathbb{N} = \mu\Phi$ where

$$\begin{aligned}\Phi(X) &= \{0\} \cup \{x+1 \mid x \in X\} \\ &= \{y \mid y = 0 \vee \exists x (y = x + 1 \wedge x \in X)\}\end{aligned}$$

The functor associated with Φ is obtained by removing all first-order parts from Φ : $\varphi(\alpha) = 1 + \alpha$. The initial algebra $\text{In}_\varphi : \varphi(\mu\varphi) \rightarrow \mu\varphi$ is the familiar structure of unary natural numbers $\mathbb{N} := \mu\varphi$ generated by zero and successor.

A step function $f : \varphi(\alpha) \rightarrow \alpha$ consists of $f_0 : \alpha$ and $f_1 : \alpha \rightarrow \alpha$. The iteration $g := \mathbf{It}_\varphi(f) : \mathbb{N} \rightarrow \alpha$ is defined recursively by $g(0) = f_0$, $g(S(n)) = f_1(g(n))$.

Remarks: 1. The variables x, y may range over abstract objects, for example the real numbers. 2. Category theory is only needed to explain realizability. The “user” doesn’t have to know anything about this.

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Reals as processes

We view a real number x as a *process* emitting digits that provide better and better approximations to x .

Processes are conveniently modelled by *final coalgebras*.

Realizability naturally associates final coalgebras with *coinductive definitions*, i.e. greatest fixed points of monotone predicate transformers (in the same way as it associates initial algebras with inductive definitions).

Hence, we use coinductive definitions to model a digital approach to computable analysis.

Coinduction

Coinduction is dual to induction. Given a monotone predicate transformer Φ we have a coinduction scheme for its greatest fixed point $\nu\Phi$:

$$\mathcal{P} \subseteq \Phi(\mathcal{P}) \rightarrow \mathcal{P} \subseteq \nu\Phi$$

The associated data type is the final coalgebra

$$\text{Out}_\varphi : \nu\varphi \rightarrow \varphi(\mu\varphi).$$

The coinduction scheme is realized by the coiterator \mathbf{Coit}_φ that coiterates any “step function” (i.e. φ -coalgebra) $f : \alpha \rightarrow \varphi(\alpha)$ to a coalgebra morphism $\mathbf{Coit}_\varphi(f) : \alpha \rightarrow \mu\varphi$ with computation rule (i.e. morphism equation)

$$\text{Out}_\varphi(\mathbf{Coit}_\varphi(f)(a)) = \mathbf{map}_\varphi(\mathbf{Coit}_\varphi(f))(f(a))$$

Equivalently, using the fact that Out_φ has an inverse In_φ ,

$$\mathbf{Coit}_\varphi(f)(a) = \text{In}_\varphi(\mathbf{map}_\varphi(\mathbf{Coit}_\varphi(f))(f(a)))$$

Example: Signed digit representation

We are after a signed digit representation of real numbers x in the compact interval $\mathbb{I} := [-1, 1]$, i.e. we want

$$x = \sum_{n=0}^{\infty} d_n \cdot 2^{-(n+1)} \quad (1)$$

where $d_i \in \text{SD} := \{-1, 0, 1\}$.

(1) is equivalent to the fact that there are $x_0, x_1, \dots \in \mathbb{I}$ such that $x = 1/2(d_0 + x_0) = 1/2(d_0 + 1/2(d_1 + x_1)) = \dots$

This suggests the following coinductive predicate on \mathbb{I} :

$$C_0 = \nu X. \{x \mid \exists d \in \text{SD} \exists x_0 (x = \frac{d + x_0}{2} \wedge X(x_0))\}$$

The data type associated with C_0 is the type of infinite streams of signed digits. A stream d_0, d_1, \dots realizes $C_0(x)$ precisely when (1) holds.

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Extracting exact real number algorithms

Using coinduction one can prove, for example:

Theorem 1 $x \in C_0$ iff $\forall n \in \mathbb{N} \exists q \in \mathbb{Q} \cap \mathbb{I} |x - q| \leq 2^{-n}$.

Theorem 2 If $x, y \in C_0$ then $\frac{x+y}{2} \in C_0$.

Theorem 3 If $x, y \in C_0$ then $xy \in C_0$.

From the proofs of these theorems one extracts programs translating between the signed-digit- and the Cauchy-representation, as well as implementations of addition and multiplication w.r.t. the signed digit representation.

Similar implementations were studied by Edalat, Potts, Heckmann, Escardo, Marcial-Romero, Ciaffaglione, Gianantonio, ...

The difference is that we *extract* the programs, together with their correctness proofs.

Characterizing uniform continuity by induction/coinduction

Recall the coinductive definition of reals in \mathbb{I} that have a signed digit representation:

$$C_0 = \nu X. \{x \mid \exists d \in \text{SD} \exists x_0 (x = \text{av}_d(x_0) \wedge X(x_0))\}$$

where $\text{av}_d(x_0) := \frac{d+x_0}{2}$.

We generalize this to a characterization of (uniformly) continuous functions $f : \mathbb{I} \rightarrow \mathbb{I}$:

$$C_1 = \nu X. \mu Y. \{f \mid \exists d \in \text{SD} \exists f_0 (f = \text{av}_d \circ f_0 \wedge X(f_0)) \\ \vee \forall d \in \text{SD} Y(f \circ \text{av}_d)\}$$

The left disjunct is analogous to C_0 and means that f *emits* a digit.

The right disjunct means that f *absorbs* a digit from the input.

Memo tries for continuous functions

Theorem 4 $f \in \mathbb{I}$ is continuous iff $f \in C_1$.

From the proof of this theorem one extracts programs translating between realisers of “ f is continuous” (where continuity has to be defined in a constructively meaningful way) and realisers of “ $f \in C_1$ ”.

What is a realiser of “ $f \in C_1$ ”?

It is a finitely branching non-wellfounded tree describing when f emits and absorbs digits. I.p. it is a *data structure*, not a function.

Similar trees have been studied by P. Hancock, D. Pattinson, N. Ghani.

Extracting memoized exact real arithmetic

The definition of $C_1 \subseteq \mathbb{I}^{\mathbb{I}}$ can be generalised to $C_n \subseteq \mathbb{I}^{(\mathbb{I}^n)}$.

Theorem 5 The average function lies in C_2 .

Theorem 6 Multiplication lies in C_2 .

From Theorems 5,6 one extracts implementations of addition and multiplication as memo-tries.

Experiments show considerable speed-up when sampling “hard” functions (e.g. high iterations of the logistic map) on a very fine grid.

Theorem 7 If $f \in C_1$, then $\int f \in C_0$.

The extracted program program has some similarity with A. Simpson's, but is more efficient because the functions to be integrated are represented differently.

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Conclusion (program extraction)

Program extraction replaces the traditional workflow

specification \rightarrow coding \rightarrow verification

by

specification \rightarrow proof \rightarrow program extraction

The advantages are

- ▶ proofs can be checked automatically for correctness
- ▶ program extraction is automatic and includes a correctness proof
- ▶ Realizability extends the original specification of the main program to specifications of all sub-programs. This is important when sub-programs are optimized.

Results, open questions

- ▶ Program extraction turns out to be very helpful (not a burden) in the example areas covered.
- ▶ New (correct!) programs have been extracted that would have been difficult to “guess”.
- ▶ Using a fine tuning of realisability it is possible to do abstract mathematics as usual, and still get computational content.
- ▶ For example, there is no problem with using discontinuous and partial functions (sign function, least root of a polynomial).

- ▶ Can we apply program extraction to areas that are less mathematical in nature?
- ▶ Can we address resource issues?

Spin-off

The proof-as-programs paradigm is not only useful for program extraction, but also creates new ideas, methods and results.

For example:

- ▶ new methods and results in approximation- fixedpoint- and ergodic-theory
- ▶ Memoized computation in higher types
- ▶ New forms of bar recursion
- ▶ Selection functions
- ▶ New “computationally efficient” definitions of uniform continuity
- ▶ Uniform logical connectives
- ▶ A new “ultra memoized” model and implementation of computation in higher types

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