# The Weihrauch Degree of Finding Nash Equilibria in Multiplayer Games

Tonicha Crook
Joint work with Arno Pauly

Swansea University

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# Outline

# Introduction

Algorithms to find Nash equilibria have already been investigated if the payoffs in our games are given as integers. However it has not been for real numbers.

We will explore how non-computable the task of finding Nash equilbria is using Weihrauch reducibility.

For one or two player games, a complete classification has already been obtained but we are addressing the situation with multiplayer games.

# Game Theory

## Nash Equilibirum

A strategy vector $s^* = (s_1^* \dots s_n^*)$ is a *Nash Equilibrium* if for each player $i \in N$ and each strategy $s_i \in S_i$, $u_i(s^*) \geq u_i(s_i, S_i^*)$ is satisfied.

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|---|---|---|---|
| Player I | $A_I$ | 0,1 | 1,0 |
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Figure: Simple Game in Strategic Form

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# Real Number Computability

## Computable

A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is *computable*, if there is a computable function $F ::\subseteq \sum^{\mathbb{N}} \to \sum^{\mathbb{N}}$ such that $F(p)$ is a decimal expansion of $f(x)$ whenever $p$ is a decimal expansion of $x$.

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## Computable (final)

A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is *computable*, if there is a computable function $F$ such that $\rho(F(p)) = f(\rho(p))$. Where $\rho(q) = x$ is a sequence $(q_n)_n \in \mathbb{N} \in \mathbb{Q}^{\mathbb{N}}$ representing $x \in \mathbb{R}$, if $|x - q_n| < 2^{-n}$ for all $n \in \mathbb{N}$.

# Computable Analysis and Weihrauch Reducibility

## Represented Spaces

A *represented space* $(X, \delta)$ is a set $X$ together with a surjective partial function $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

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A multivalued function between represented spaces is Weihrauch reducible to another, if there is an otherwise computable procedure invoking the second multi-valued function as an oracle exactly once that solves the first.

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## Weihrauch Degree

The Weihrauch degrees are the equivalence classes for Weihrauch reductions.

**Parallel**

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### Sequential

The degree $f^\diamond$ represents being allowed to invoke $f$ any finite number of times (not specified in advance), where later queries can be computed from previous answers.

# Our Results

## Nash

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## All-or-Unique Choice

For a represented space $X = \{0,1\}^{\mathbb{N}}$, denote a multivalued function $\mathrm{AoUC}_{[0,1]} :\subseteq \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}$ via $\{A \in \mathcal{A}((X)| \ |A| = 1\} \cup \{X\}$

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## BRoot

$\mathrm{BRoot} \subseteq: \mathbb{R}[X] \rightrightarrows [0,1]$ map real polynomials to a root in $[0,1]$, provided there is one.

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**Main Theorem**

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**Theorem**

$\mathrm{AoUC}^*_{[0,1]} \equiv_W \mathrm{BRoot}^*$

# Our Results

**Corollary**

*Let* $f : \mathbf{X} \to \mathbf{Y}$ *be a function where* $\mathbf{Y}$ *is computably admissible. Then if* $f \leq_W \mathrm{Nash}$ *then* $f$ *is already computable.*

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**Corollary**

*Nash is Las Vegas computable.*

**Corollary**

*Nash is Monte Carlo computable and we can compute a postive lower bound for the success chance from the dimensions of the game.*

Thank You
Any Questions?