Games with Counters

Patrick Totzke

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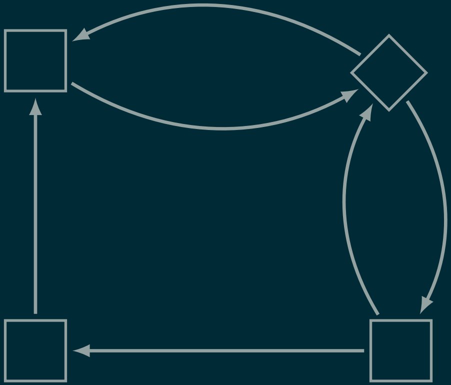
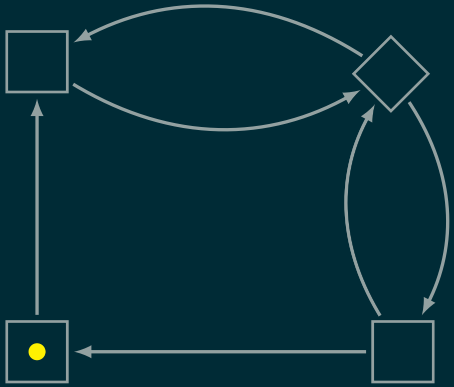
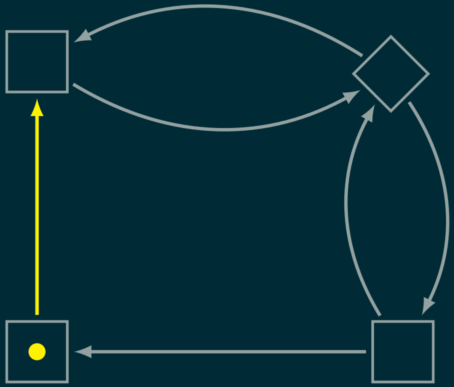
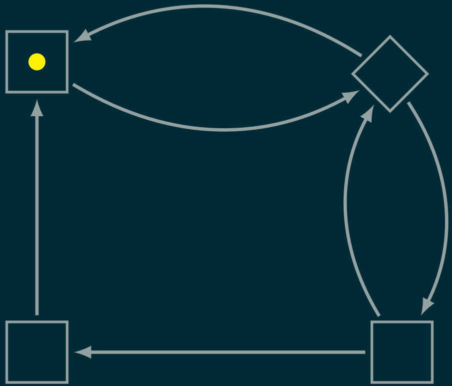
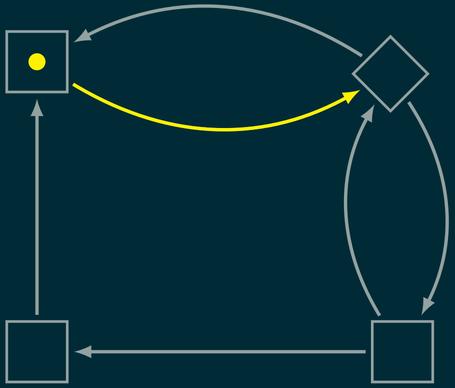
BCTCS – April 7, 2020

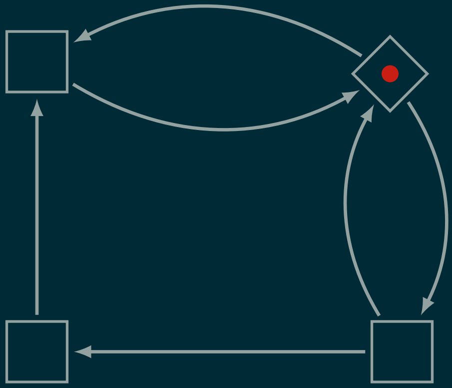


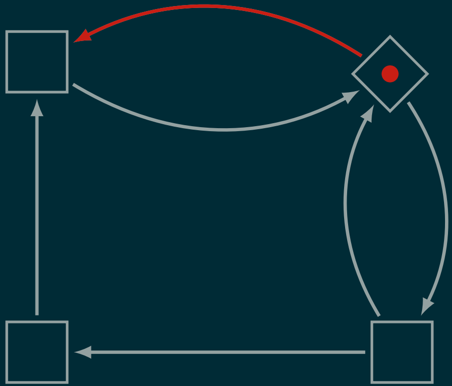


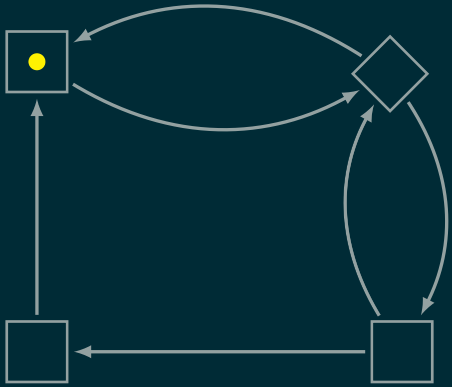


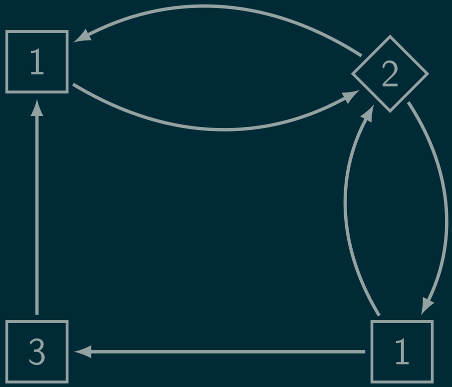








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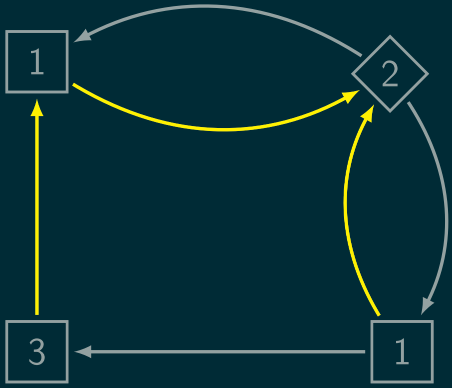
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Parity Games

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Winning Condition: □ wins if the maximal colour seen infinitely often is even.

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Interesting because: PG solving is equivalent to model checking for modal-μ which supersedes the most common temporal specification languages (LTL, CTL, ...)

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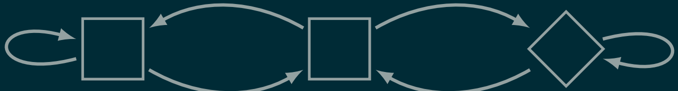

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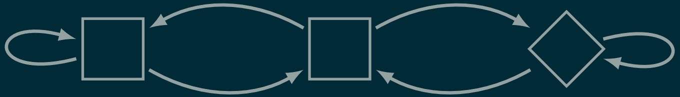

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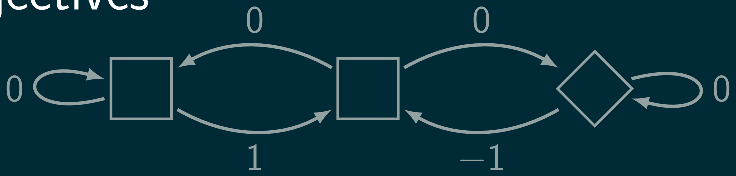

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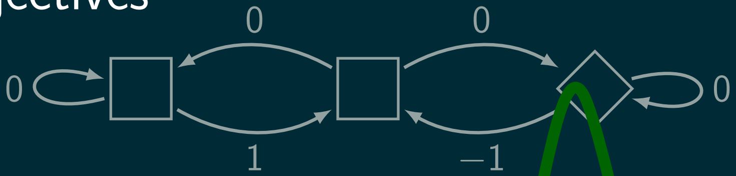

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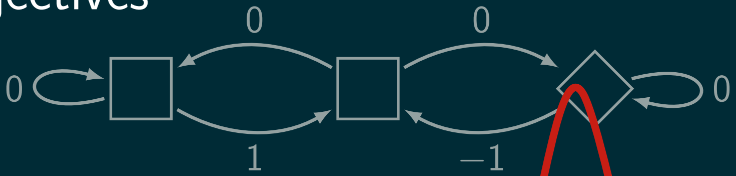

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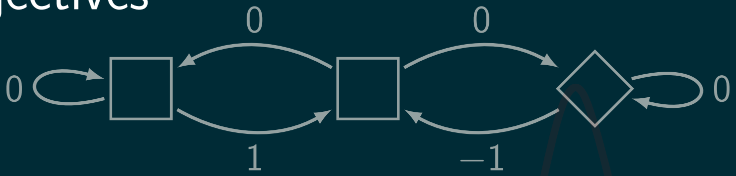

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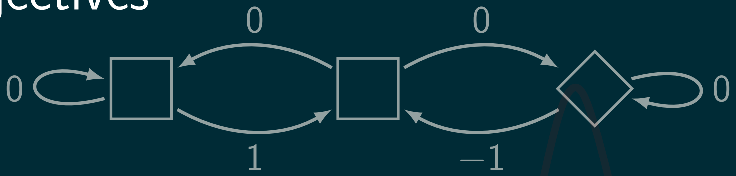

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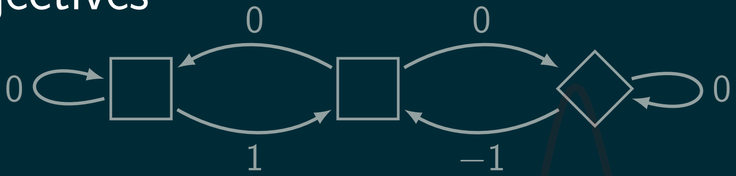

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All these enjoy positional determinacy!

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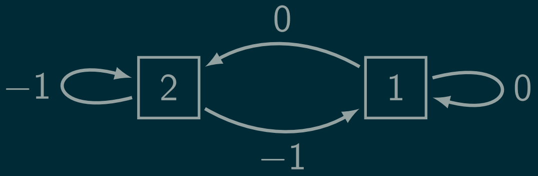

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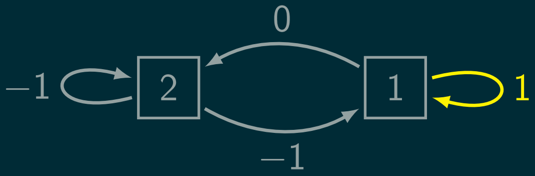

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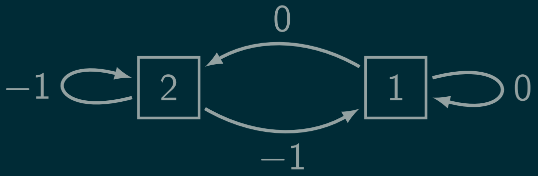

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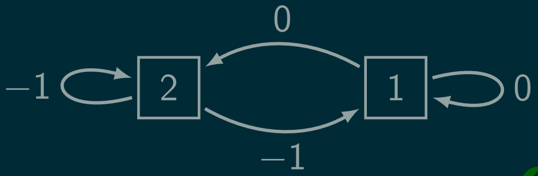

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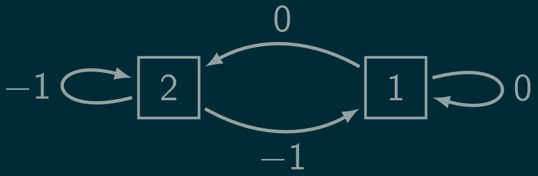

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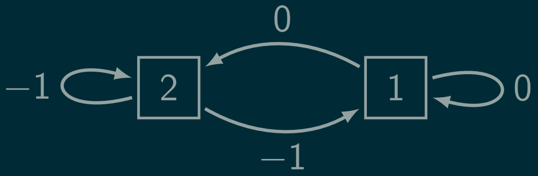

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All positionally determined and undecidable!

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Famously decidable

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... and EXPSPACE-complete if n_0 is existentially quantified.

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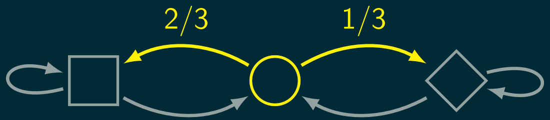

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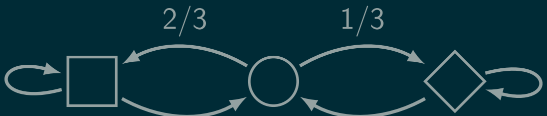

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- A *cylinder* is a set wV^ω of runs that share the prefix $w \in V^*$.
- Define a measure $\mathbb{P}_s^{\sigma,\tau}$ on cylinders by $\mathbb{P}_s^{\sigma,\tau}(sV^\omega) = 1$ and

$$\mathbb{P}_s^{\sigma,\tau}(wstV^\omega) = \begin{cases} \mathbb{P}_s^{\sigma,\tau}(wsV^\omega) \cdot \sigma(ws)(t) & \text{for } s \in V_\Diamond \\ \mathbb{P}_s^{\sigma,\tau}(wsV^\omega) \cdot \tau(ws)(t) & \text{for } s \in V_\Box \\ \mathbb{P}_s^{\sigma,\tau}(wsV^\omega) \cdot \delta(ws)(t) & \text{for } s \in V_\bigcirc \end{cases}$$

- Caratheodory's theorem guarantees a unique extension of $\mathbb{P}_s^{\sigma,\tau}$ to $\mathcal{O} \stackrel{\text{def}}{=}$ the Borel sigma-algebra generated by all cylinders.

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Max/Min choose their strategies to maximize/minimize $\mathbb{P}_s^{\sigma,\tau}(Obj)$.

Objectives – Values and Optimality

Weak Determinacy: For all Borel Objectives *Obj* and countable (even concurrent) games

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Stochastic Energy∩Parity Games

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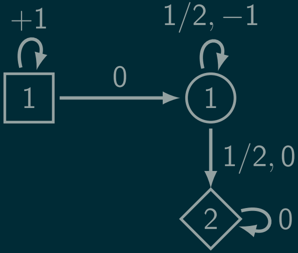

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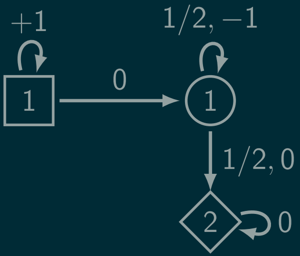
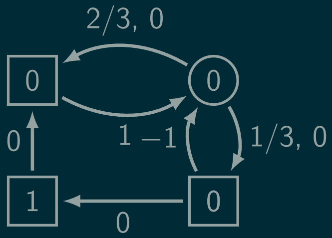


If they do they may require infinite memory:



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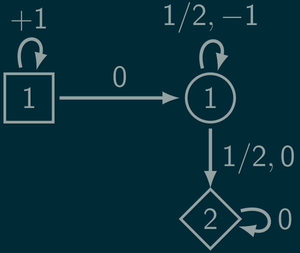

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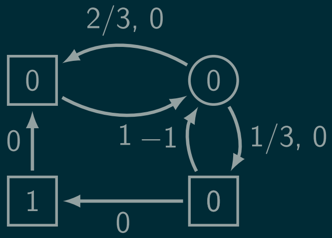

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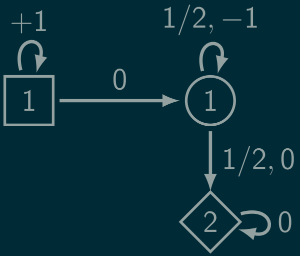

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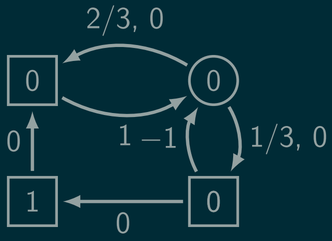

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A Step Back: Parity Games are..

- turn-based
- zero-sum
- perfect information
- ω-regular winning conditions
- played on finite graphs
- non-stochastic

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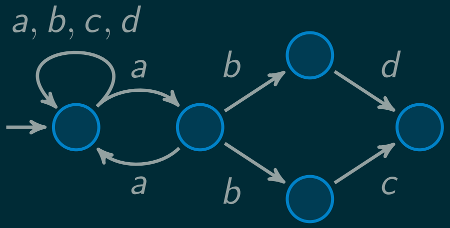
- pick actions such that $\mathcal{P}(\mathit{init} \rightsquigarrow \mathit{final}) = 1$



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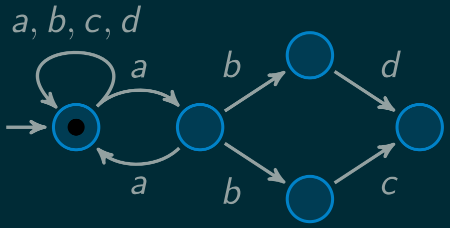

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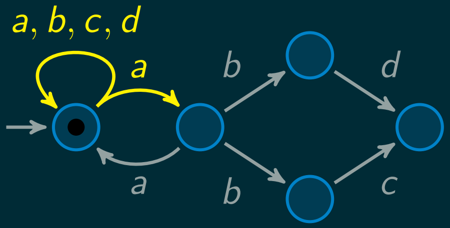

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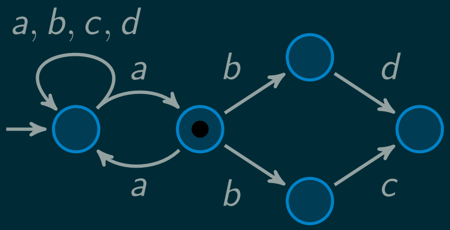

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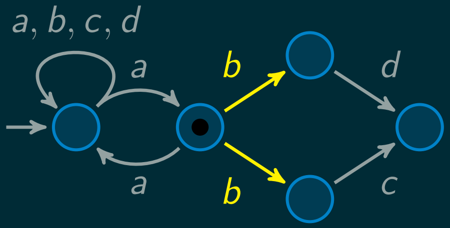

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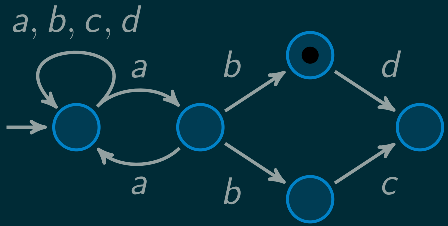

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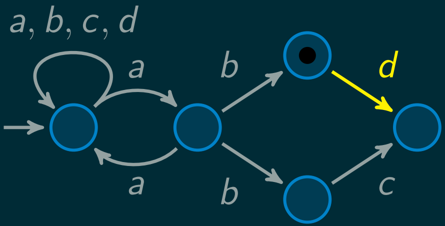

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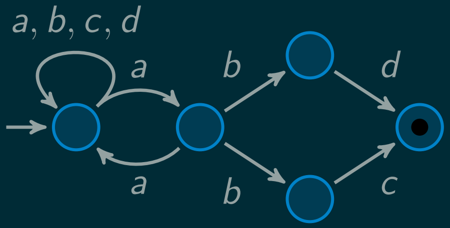

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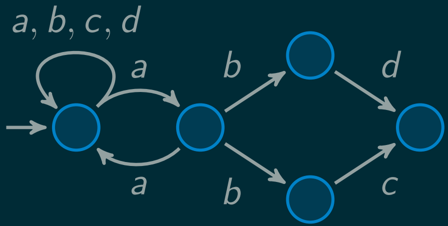

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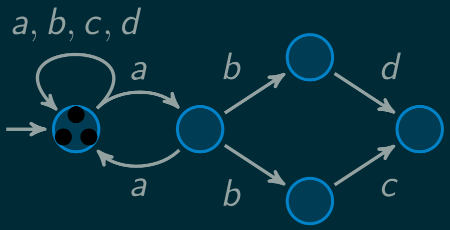

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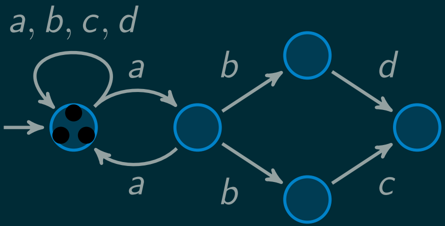

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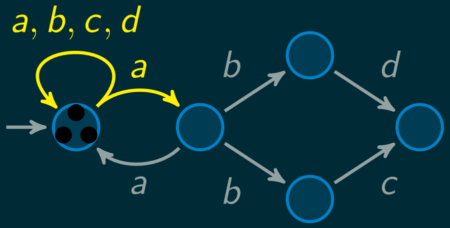

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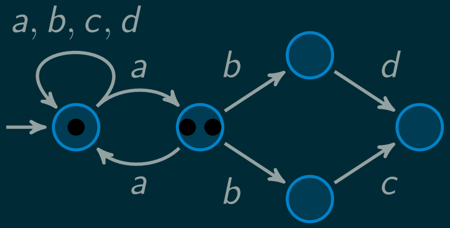

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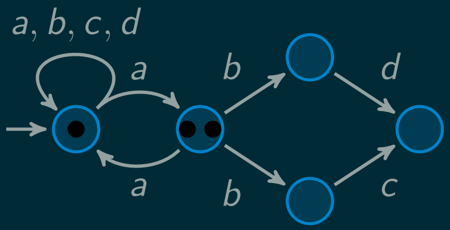

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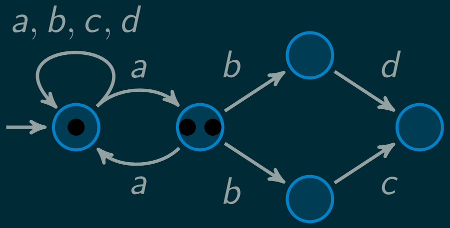

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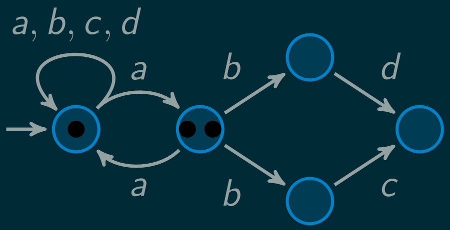

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Question: Can I synchronize M^n for every n? Construct strategy $\sigma : \mathbb{N}^Q \to Act$? (Decidable and EXP-hard)

References

[1] R. Niskanen, I. Potapov, and J. Reichert. "On decidability and complexity of low-dimensional robot games". In: 107 (2020), pp. 124–141.

[2] R. Mayr et al. "MDPs with Energy-Parity Objectives". In: *LICS* (2017).

[3] M. Jurdziński, R. Lazić, and S. Schmitz. "Fixed-Dimensional Energy Games are in Pseudo-Polynomial Time". In: *ICALP*. 2015, pp. 260–272.

[4] P. A. Abdulla et al. "Infinite-State Energy Games". In: *LICS*. 2014.

[5] S. Böhm, S. Göller, and P. Jancar. "Bisimulation equivalence and regularity for real-time one-counter automata". In: 80.4 (2014), pp. 720–743.

[6] K. Chatterjee and L. Doyen. "Energy parity games". In: *Theor Comput Sci* 458 (2012), pp. 49–60.

[7] O. Serre. "Parity Games Played on Transition Graphs of One-Counter Processes". In: *FoSSaCS*. 2006.

[8] A. Maitra and W. Sudderth. "Stochastic games with Borel payoffs". In: *Stochastic Games and Applications*. Ed. by A. Neyman and S. Sorin. Kluwer, Dordrecht, 2003, pp. 367–373.

[9] G. Sénizergues. "Decidability of Bisimulation Equivalence for Equational Graphs of Finite Out-Degree". In: *FOCS*. IEEE Computer Society, 1998, pp. 120–129.

[10] I. Walukiewicz. "Pushdown Processes: Games and Model Checking". In: *CAV*. 1996.

[11] M. L. Puterman. *Markov Decision Processes*. 1994.