Towards a trichotomy for quantified H-coloring

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CiE 2006, Swansea

The *Constraint Satisfaction Problem* CSP may be defined to have

- Input: a structure $\mathcal{T}$ and a sentence $\varphi := \exists \mathbf{x} Q(\mathbf{x})$, where Q is a conjunction of positive atoms.
- Question: does $\mathcal{T} \models \varphi$?

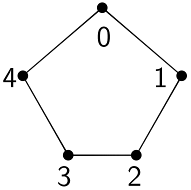
In fact, we will only be concerned with the parameterisation of this problem with respect to the template $\mathcal{T}$.

We define the *non-uniform* constraint satisfaction problem $\mathrm{CSP}(\mathcal{T})$ to have

- Input: a sentence $\varphi := \exists \mathbf{x} Q(\mathbf{x})$, where Q is a conjunction of positive atoms.
- Question: does $\mathcal{T} \models \varphi$?

The problems $\mathrm{CSP}(\mathcal{T})$ are readily seen to be in NP. Furthermore, many natural problems in NP readily translate to problems $\mathrm{CSP}(\mathcal{T})$, for some template $\mathcal{T}$. Examples include *graph 2-colourability*, which is tractable, and 3-*satisfiability* and *graph 3-colourability*, which are NP-complete.

The problem 3-*colourability* translates to the problem $\mathrm{CSP}(\mathcal{K}_3)$, where $\mathcal{K}_3$ is the antireflexive 3-clique (triangle).
Given an input graph $\mathcal{G}$, that we want to test for 3-colourability, we construct an input $\varphi_{\mathcal{G}}$ which will be the existential quantification of a conjunction of the facts of $\mathcal{G}$ (sometimes called the canonical query of $\mathcal{G}$).
For example, let $\mathcal{G}$ be the 5-cycle (pentagon).



then $\varphi_{\mathcal{G}}$ is

$$\begin{array}{ll}\exists x_0, x_1, x_2, x_3, x_4 & E(x_0, x_1) \wedge E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_4) \wedge E(x_4, x_0) \\ & E(x_1, x_0) \wedge E(x_2, x_1) \wedge E(x_3, x_2) \wedge E(x_4, x_3) \wedge E(x_0, x_4)\end{array}$$

Given the breadth of problems $\mathrm{CSP}(\mathcal{T})$, it is tempting to view it as a microchosm of all NP. However, it has been conjectured[1] that the problems $\mathrm{CSP}(\mathcal{T})$ exhibit *dichotomy*, that is they are either tractable of NP-complete, depending on $\mathcal{T}$. In contrast, NP is known[2] not to exhibit such a dichotomy (unless it actually coincides with P).

Much effort has gone into the complexity classification problem for $\mathrm{CSP}(\mathcal{T})$, and many partial dichotomy results are known.

[1] Feder/Vardi 99, Bulatov/Krokhin/Jeavons 00.

[2] Ladner 75.

For example:

- (Schaefer 1978) Dichotomy when $||\mathcal{T}|| \leq 2$ (i.e. on boolean domains); recently extended to
- (Bulatov 2002) Dichotomy when $||\mathcal{T}|| \leq 3$.
- (Hell/Nešetřil 1990) Dichotomy when $\mathcal{T}$ ranges over undirected graphs.

  Specifically: if $\mathcal{T}$ has a self-loop or is bipartite, then $\mathrm{CSP}(\mathcal{T})$ is tractable, otherwise it is NP-complete.

Schaefer's result is obtained through an exhaustive analysis of the types of relations that may constitute $\mathcal{T}$. Bulatov's result is similarly obtained, but with much machinery from universal algebra. Hell and Nešetřil's result is obtained by combinatorial arguments and contradiction – in contrast to the others, it is non-constructive.

Recall that $\mathrm{CSP}(\mathcal{T})$ is the problem with

- Input: a sentence $\varphi := \exists \mathbf{x} Q(\mathbf{x})$, where Q is a conjunction of positive atoms.
- Question: does $\mathcal{T} \models \varphi$?

It now makes sense to define the *Quantified Constraint Satisfaction Problem* $\mathrm{QCSP}(\mathcal{T})$ with

- Input: a sentence ψ of the form

$$\forall \mathbf{x_1} \exists \mathbf{x_2} \forall \mathbf{x_3} \exists \mathbf{x_4} \dots \forall \mathbf{x_{2n+1}} \exists \mathbf{x_{2n+2}} Q(\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_{2n+2}})$$

  (for some $n \geq 1$), where Q is a conjunction of positive atoms.
- Question: does $\mathcal{T} \models \psi$?

$\mathrm{QCSP}(\mathcal{T})$ is always in Pspace, and problems such as *quantified 3-satisfiability* (which is Pspace-complete) may be readily translated into its framework.

We will be concerned with the complexity of $\mathrm{QCSP}(\mathcal{T})$ when $\mathcal{T}$ is any *undirected, antireflexive* graph.

It is known[3] that the *quantified k-colouring problem*, $\mathrm{QCSP}(\mathcal{K}_k)$, is tractable for $k = 1, 2$ and Pspace-complete for $k \geq 3$.

[3] Borner/Krokhin/Bulatov/Jeavons 02.

We aim to prove that for a large class of templates $\mathcal{T}$, an input ψ of the form

$$\forall \mathbf{x_1} \exists \mathbf{x_2} \forall \mathbf{x_3} \exists \mathbf{x_4} \ldots \forall \mathbf{x_{2n+1}} \exists \mathbf{x_{2n+2}} Q(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{2n+2}})$$

is either readily seen to be false on $\mathcal{T}$, or is logically equivalent to an input ψ' of the form (for some m)

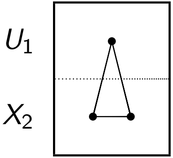
$$\forall x_1^1 \exists \mathbf{x_2^1} Q_1(x_1^1, \mathbf{x_2^1}) \ \wedge \ \ldots \ \wedge \ \forall x_1^m \exists \mathbf{x_2^m} Q_m(x_1^m, \mathbf{x_2^m})$$

Note that, since $x_1^1, \ldots, x_1^m$ are not tuples but single variables, the evaluation of ψ' on $\mathcal{T}$ is at worst in NP.

Henceforth, all sentences will be assumed to be quantified conjunctive positive. We prefer to visualise such sentences via a graphical representation. For example, we think of the sentence ψ_0:

$$\forall x \exists y \exists z \; E(x,y) \wedge E(y,x) \wedge E(y,z) \wedge E(z,y) \wedge E(z,x) \wedge E(x,z)$$

as the *partitioned graph* $\mathfrak{P}_{\psi_0}$:



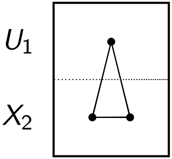
Truth of a (q.c.p.) sentence ψ on a structure $\mathcal{T}$ is best understood in terms of a game on $(\mathfrak{P}_\psi, \mathcal{T})$. In the game, an *adversary* plays the universal partitions and a *prover* plays the existential partitions; the prover wins if the resultant map is a homomorphism. This game is exactly the model-checking (or Hintikka) game of ψ on $\mathcal{T}$.

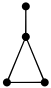
Examples

We have already met the sentence ψ_0:

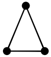
$$\forall x \exists y \exists z\ E(x,y) \wedge E(y,x) \wedge E(y,z) \wedge E(z,y) \wedge E(z,x) \wedge E(x,z)$$

and its associated partitioned graph $\mathfrak{P}_{\psi_0}$:





$\mathcal{T}_1$

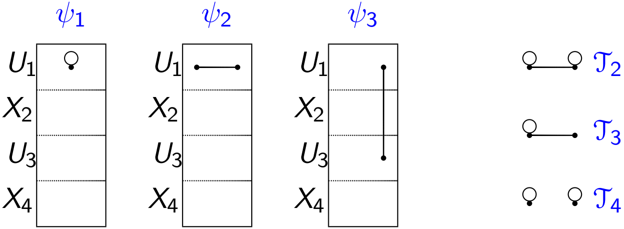


$\mathcal{T}_0$

We have $\mathcal{T}_0 \models \psi_0$ but $\mathcal{T}_1 \not\models \psi_0$.

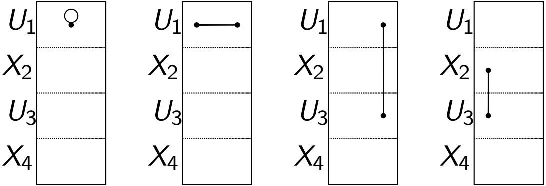
Easy Cases

- If $\mathcal{T}$ is a reflexive clique then, for all ψ, $\mathcal{T} \models \psi$.
- If $\mathcal{T}$ is not a reflexive clique then: if there is an edge between some distinct
  $x \in U_i$ and $y \in U_j$ (any i, j) then $\mathcal{T} \not\models \psi$.



$\mathcal{T}_2 \models \psi_1, \psi_2, \psi_3$; $\mathcal{T}_3 \not\models \psi_1, \psi_2, \psi_3$; and $\mathcal{T}_4 \models \psi_1$ but $\mathcal{T}_4 \not\models \psi_2, \psi_3$.

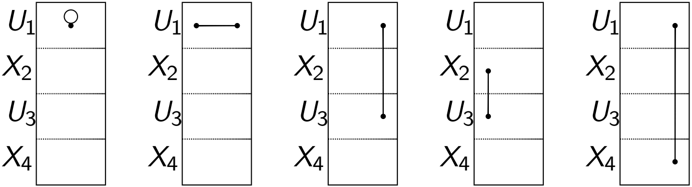
- If $\mathcal{T}$ is antireflexive then, if there is an edge between some $x \in U_i$ and $y \in U_j$ (any i, j) or $x \in X_i$ and $y \in U_j$ (for $i < j$) then $\mathcal{T} \not\models \psi$. We have the following edge types forbidden:



- If $\mathcal{T}$ has an isolated vertex then, if there is an edge between some

  $$x \in U_i \text{ and } y \in U_j \text{ (any } i,j) \ \underline{\text{or}}\ x \in E_i \text{ and } y \in U_j \text{ (any } i,j)$$

  then $\mathcal{T} \not\models \psi$. We have the following edge types forbidden:



In this case, all edges other than those between existential partitions, are forbidden. It follows that any vertices in universal partitions are isolated, and that $\mathrm{QCSP}(\mathcal{T})$ is logspace equivalent to $\mathrm{CSP}(\mathcal{T})$.

Returning to the logical formulation: if $\mathcal{T}$ is a graph with an isolated vertex, then any input ψ for $\mathrm{QCSP}(\mathcal{T})$ of the form

$$\forall \mathbf{x_1} \exists \mathbf{x_2} \forall \mathbf{x_3} \exists \mathbf{x_4} \ldots \forall \mathbf{x_{2n+1}} \exists \mathbf{x_{2n+2}} Q(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{2n+2}})$$

is either readily seen to be false on $\mathcal{T}$, or is actually logically equivalent to $\psi' :=$

$$\exists \mathbf{x_2} \ \exists \mathbf{x_4} \ \ldots \ \exists \mathbf{x_{2n+2}} \ Q(\mathbf{x_2}, \mathbf{x_4}, \ldots, \mathbf{x_{2n+2}})$$

which is an input for $\mathrm{CSP}(\mathcal{T})$.

Examples

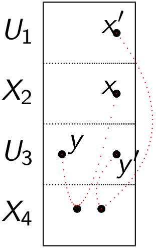
It follows immediately that
$\mathrm{QCSP}(\mathcal{K}_2 \uplus \mathcal{K}_1)$ is tractable, and
$\mathrm{QCSP}(\mathcal{K}_3 \uplus \mathcal{K}_1)$ is NP-complete.

Non-connected templates

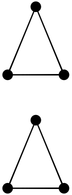
- Let $\mathcal{T}$ be a non-connected graph. If there is a path in $\mathfrak{P}_\psi$ between some

  $$x \in X_i \text{ and } y \in U_j \text{ (for } i < j\text{) } \underline{\text{or}}\ x \in U_i \text{ and } y \in U_j \text{ (any } i, j\text{)}$$

  then $\mathcal{T} \not\models \psi$.



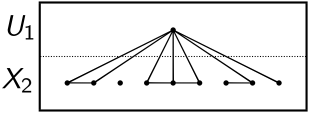
$\mathcal{T} := \mathcal{K}_3 \uplus \mathcal{K}_3$



Adversary may always win by playing y [y'] in a different component of $\mathcal{T}$ from x [x'].

If $\mathfrak{P}_\psi$ has no path between any $x \in X_i$ and $y \in U_j$ (for $i < j$), then $\mathfrak{P}_\psi$ is readily seen to be equivalent to the 'Π_2' partitioned graph $\mathfrak{P}'$ obtained by collapsing all universal partitions to U_1 and all existential partitions to X_2.

Furthermore, if $\mathfrak{P}_\psi$ also has no path between any $x \in U_i$ and $y \in U_j$ (any i, j), then this collapsed $\mathfrak{P}'$ will be the disjoint union of partitioned graphs $\mathfrak{P}_i$ in Π_2-*fan form*. Let ψ_i be the sentence of which $\mathfrak{P}_i$ is the graphical representation.



Note that we can always establish in NPwhether $\mathcal{T} \models \psi_i$. It follows that, when $\mathcal{T}$ is non-connected, $\mathrm{QCSP}(\mathcal{T})$ is in NP.

Example

$\mathrm{QCSP}(\mathcal{K}_3 \uplus \mathcal{K}_3)$ is NP-complete.

Bipartite templates

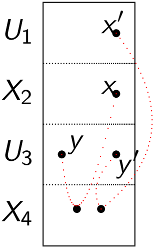
Bipartite templates behave similarly to non-connected templates.

- Let $\mathcal{T}$ be a bipartite graph. If there is a path in $\mathfrak{P}_\psi$ between some

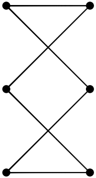
  $$x \in X_i \text{ and } y \in U_j \text{ (for } i < j) \ \underline{\text{or}}\ x \in U_i \text{ and } y \in U_j \text{ (any } i, j)$$

  then $\mathcal{T} \not\models \psi$.

Recall that in a bipartite graph there may be no odd path from one ‘part’ to itself, and no even path from one ‘part’ to the other.



$\mathfrak{T} := \mathcal{C}_6$



If the path in $\mathfrak{P}_\psi$ is of odd length, then adversary may win by playing y [y'] in the same ‘part’ of $\mathfrak{T}$ as x [x'].
If the path in $\mathfrak{P}_\psi$ is of even length, then adversary may win by playing y [y'] in the opposite ‘part’ of $\mathfrak{T}$ from x [x'].

It follows that any input ψ is either readily seen not to be true on bipartite $\mathcal{T}$, or to be equivalent to some disjoint union of sentences ψ_i whose partitioned graphs $\mathfrak{P}_{\psi_i}$ are in Π_2-*fan form*.

Tractability of $\mathrm{QCSP}(\mathcal{T})$, when $\mathcal{T}$ is bipartite, follows relatively straightforwardly.

Are there other templates for which partitioned graphs collapse?

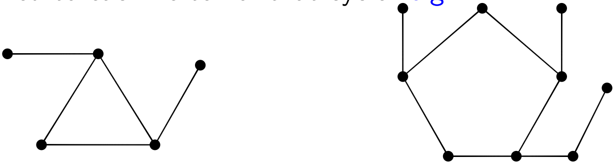
We can not use the same forbidden paths method on any other undirected templates. This is because, for any connected non-bipartite graph $\mathcal{T}$, there exists a number M s.t.

> for all $x, y \in V(\mathcal{T})$ and for all $m \geq M$ there is an m-path between x and y.

We conjecture that $\mathrm{QCSP}(\mathcal{T})$ is Pspace-complete for all connected non-bipartite $\mathcal{T}$.

Pspace-completeness results

We define an Odd Catherine Wheel (OCW) to be an odd cycle with a path affixed to each vertex of that cycle. e.g.



We can prove that, if $\mathcal{T}$ is an OCW, then $\mathrm{QCSP}(\mathcal{T})$ is Pspace-complete. If $\mathcal{T}$ is based on the $(2k+1)$-cycle, we prove it by direct reduction from the problem $\mathrm{QCSP}(\mathcal{B}^{2k+1}_{NAE})$ (where $\mathcal{B}^{2k+1}_{NAE}$ is the boolean structure with a single $2k+1$-ary not-all-equal relation).

A limited trichotomy

Noting that a connected non-bipartite graph with a unique cycle is necessarily an OCW, we deduce the following:

Let $\mathcal{T}$ range over antireflexive, undirected graphs with at most one cycle. Then $\mathrm{QCSP}(\mathcal{T})$ is

- tractable, if $\mathcal{T}$ is bipartite,
- NP-complete, if $\mathcal{T}$ is non-connected and non-bipartite, and
- Pspace-complete otherwise.