EXACT DICKSON ORDINALS

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Abstract.

We argue classicaly. We identify a natural number with the set of its predecessors.

Definition 0.1. *Let* $n \in \mathbb{N}$, $\vec{x}, \vec{y} \in \mathbb{N}^n$ *and* $M, N \subseteq \mathbb{N}^n$.

$$\begin{aligned} \vec{x} \leq \vec{y} &:\Leftrightarrow \forall i < n . x_i \leq y_i \\ \vec{x} < \vec{y} &:\Leftrightarrow \vec{x} \leq \vec{y} \wedge \vec{x} \neq \vec{y} \\ M^* &:= \{\vec{y} \in \mathbb{N} | \forall \vec{x} \in M . \vec{x} \not\leq \vec{y}\} \\ N \sqsubseteq M &:\Leftrightarrow N^* \subseteq M^* \\ N \sqsubset M &:\Leftrightarrow N \sqsubseteq M \wedge N \neq M \end{aligned}$$

Proposition 0.1. 1. *The relation* $\leq$ *is reflexive, transitive and antisymmetric.*

2. *The relation* $\sqsubseteq$ *is reflexive and transitive.*

Proof. Simple verification. ⊣

Proposition 0.2. *Every non empty set* $M \subseteq \mathbb{N}^n$ *has an* $<$*-minimal element.*

Proof. Natural induction. ⊣

Proposition 0.3.

$$N^* \subseteq M^* \Leftrightarrow \forall \vec{x} \in M . \exists \vec{y} \in N . \vec{y} \leq \vec{x}.$$

Proof. '$\Rightarrow$': Let $\vec{x} \in M$. We have $\vec{x} \leq \vec{x}$. Therefore $\vec{x} \in \complement M^* \subseteq \complement N^*$. It follows $\exists \vec{y} \in N . \vec{y} \leq \vec{x}$. '$\Leftarrow$': Let $\vec{x} \in \complement M^*$. It follows that there is an $\vec{x} \in M$ with $\vec{x} \leq \vec{z}$. Further there is a $\vec{y} \in N$ with $\vec{y} \leq \vec{x} \leq \vec{z}$. It follows $\vec{z} \in \complement N^*$. ⊣

Definition 0.2.

$$\mathsf{Incomp}_n := \{M \subseteq \mathbb{N}^n | \forall \vec{x}, \vec{y} \in M . \vec{x} \not< \vec{y} \wedge \vec{y} \not< \vec{x}\}$$

Proposition 0.4. *The relation* $\sqsubseteq$ *is reflexive, transitive and antisymmetric on* Incomp_n.

Proof. Antisymmetric: Let $M, N \in \mathsf{Incomp}_n$ with $M \sqsubseteq N$ and $N \sqsubseteq M$. By Definition we have $M^* = N^*$. Therefore also $\complement M^* = \complement N^*$. Let $\vec{x} \in M$. We have

$$\vec{x} \in M \subseteq \complement M^* = \complement N^*.$$

Therefore there is a $\vec{y} \in N$ with $\vec{y} \leq \vec{x}$. Analog it follows that there is a $\vec{z} \in M$ with $\vec{z} \leq \vec{y} \leq x$. Since $M \in \mathsf{Incomp}_n$ we have

$$\vec{x} = \vec{z} = \vec{y} \in N.$$

This shows $M \subseteq N$. That $N \subseteq M$ follows analog. ⊣

Definition 0.3. *Let $M \subseteq \mathbb{N}^n$.*

$$\mathcal{I}(M) := M \setminus \{\vec{x} \in M | \exists \vec{y} \in M. \vec{y} < \vec{x}\}.$$

Proposition 0.5. 1. $\mathcal{I}(M) \in \mathsf{Incomp}_n$
2. $\mathcal{I}(M) \subseteq M$
3. $\mathcal{I}(\mathcal{I}(M)) \subseteq \mathcal{I}(M)$.

Proof. Ad 3.:

$$\begin{aligned}\vec{x} \in \mathcal{I}(M) &\Rightarrow \vec{x} \in M \wedge \forall \vec{y} \in M. \vec{y} \not< \vec{x} \\ &\Rightarrow \vec{x} \in \mathcal{I}(M) \wedge \forall \vec{y} \in \mathcal{I}(M). \vec{y} \not< \vec{x} \quad (\text{since } \mathcal{I}(M) \subseteq M) \\ &\Rightarrow \vec{x} \in \mathcal{I}(\mathcal{I}(M)).\end{aligned}$$

⊣

Remark. *Note that*

$$M \subseteq N \not\Rightarrow \mathcal{I}(M) \subseteq \mathcal{I}(N)$$

e.g. $M = \{\langle 1, 1\rangle\}$, $N = \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$.

Proposition 0.6.

$$M^* = \mathcal{I}(M)^*$$

Proof. '$\subseteq$': Immediate since $\mathcal{I}(M) \subseteq M$.
'$\supseteq$': Let $\vec{y} \in \mathcal{I}(M)^*$ and $\vec{x} \in M$. We must show $\vec{x} \not\leq \vec{y}$. Assume $\vec{x} \leq \vec{y}$. Since $\forall \vec{x} \in \mathcal{I}(M). \vec{x} \not\leq \vec{y}$ we have $\vec{x} \notin \mathcal{I}(M)$, i.e. $\exists \vec{z} \in M. \vec{z} < \vec{x}$. Let $\vec{z}$ be minimal with this property. It follows $\vec{z} \in \mathcal{I}(M)$ and $\vec{z} < \vec{y}$ in contradiction to $\vec{y} \in \mathcal{I}(M)$. ⊣

Proposition 0.7. *Every $M \in \mathsf{Incomp}_n$ is finite.*

Proof. We show by induction on n that for every infinite sequence $(\vec{x}_i)_{i \in \mathbb{N}} \in (\mathbb{N}^n)^{\mathbb{N}}$ there are $i, j \in \mathbb{N}$ with $i < j$ and $\vec{x}_i \leq \vec{x}_j$.
The claim follows immediate for $n = 0$. Let $(\vec{x}_i)_{i \in \mathbb{N}} \in (\mathbb{N}^{n+1})^{\mathbb{N}}$. We can build a subsequence $(\vec{y}_i)_{i \in \mathbb{N}} \in (\mathbb{N}^{n+1})^{\mathbb{N}}$ with $y_{i0} \leq y_{i+10}$ for all $i \in \mathbb{N}$. By Induction Hypothesis we have $i, j \in \mathbb{N}$ with $i < j$ and $\vec{y'}_i \leq \vec{y'}_j$ where $\vec{y'}$ is $\vec{y}$ without the first component. It follows $\vec{y}_i \leq \vec{y}_j$. ⊣

We are going to define $\mathsf{ord}\, M \in \omega^n$ for non empty $M \in \mathsf{Incomp}_n$. We give the definition only for $n = 3$. It is simple to extend our work to higher dimensions.

DEFINITION 0.4. *Let $M \in \mathsf{Incomp}_3$. We define*

$$
\begin{aligned}
\bar{M}_2^0 := M_2^0 &:= \{i \in \mathbb{N} | \{i\} \times \mathbb{N} \times \mathbb{N} \subseteq M^*\} \\
\bar{M}_2^1 := M_2^1 &:= \{i \in \mathbb{N} | \mathbb{N} \times \{i\} \times \mathbb{N} \subseteq M^*\} \\
\bar{M}_2^2 := M_2^2 &:= \{i \in \mathbb{N} | \mathbb{N} \times \mathbb{N} \times \{i\} \subseteq M^*\} \\
\bar{M}_1^{0,1} &:= \{\langle i,j\rangle \in \mathbb{N}^2 | \{i\} \times \{j\} \times \mathbb{N} \subseteq M^*\} \\
\bar{M}_1^{0,2} &:= \{\langle i,j\rangle \in \mathbb{N}^2 | \{i\} \times \mathbb{N} \times \{j\} \subseteq M^*\} \\
\bar{M}_1^{1,2} &:= \{\langle i,j\rangle \in \mathbb{N}^2 | \mathbb{N} \times \{i\} \times \{j\} \subseteq M^*\} \\
M_1^{0,1} &:= \{\langle i,j\rangle \in \bar{M}_1^{0,1} | i \notin \bar{M}_2^0 \wedge j \notin \bar{M}_2^1\} \\
M_1^{0,2} &:= \{\langle i,j\rangle \in \bar{M}_1^{0,2} | i \notin \bar{M}_2^0 \wedge j \notin \bar{M}_2^2\} \\
M_1^{1,2} &:= \{\langle i,j\rangle \in \bar{M}_1^{1,2} | i \notin \bar{M}_2^1 \wedge j \notin \bar{M}_2^2\} \\
\bar{M}_0 &:= \{\langle i,j,k\rangle \in \mathbb{N}^3 | \{i\} \times \{j\} \times \{k\} \subseteq M^*\} \\
M_0 &:= \{\langle i,j,k\rangle \in \bar{M}_0 | \langle i,j\rangle \notin \bar{M}_1^{0,1} \wedge \langle i,k\rangle \notin \bar{M}_1^{0,2} \wedge \langle j,k\rangle \notin \bar{M}_1^{1,2}\}.
\end{aligned}
$$

We are going to show that the "non bar" sets in this definition are finite and that the functions giving the number of elements of these sets are computable from M. We start by showing that the membership relation for the sets in the Definition above is decidable.

PROPOSITION 0.8. *Let $M \in \mathsf{Incomp}_3$ non empty.*

1. $\{i\} \times \mathbb{N} \times \mathbb{N} \subseteq M^* \Leftrightarrow i < \min(\mathsf{pr}_0\, M)$
2. $\mathbb{N} \times \{i\} \times \mathbb{N} \subseteq M^* \Leftrightarrow i < \min(\mathsf{pr}_1\, M)$
3. $\mathbb{N} \times \mathbb{N} \times \{i\} \subseteq M^* \Leftrightarrow i < \min(\mathsf{pr}_2\, M)$
4. $\{i\} \times \{j\} \times \mathbb{N} \subseteq M^* \Leftrightarrow \forall \vec{x} \in M. i < x_0 \vee j < x_1$
5. $\{i\} \times \mathbb{N} \times \{j\} \subseteq M^* \Leftrightarrow \forall \vec{x} \in M. i < x_0 \vee j < x_2$
6. $\mathbb{N} \times \{i\} \times \{j\} \subseteq M^* \Leftrightarrow \forall \vec{x} \in M. i < x_1 \vee j < x_2$.

PROOF. Ad 1.:

$$
\begin{aligned}
\{i\} \times \mathbb{N} \times \mathbb{N} \subseteq M^* &\Leftrightarrow \forall j,k \in \mathbb{N}. \langle i,j,k\rangle \in M^* \\
&\Leftrightarrow \forall j,k \in \mathbb{N}. \forall \vec{x} \in M. \vec{x} \not\leq \langle i,j,k\rangle \\
&\Leftrightarrow \forall j,k \in \mathbb{N}. \forall \vec{x} \in M. i < x_0 \vee j < x_1 \vee k < x_2 \\
&\Leftrightarrow \forall \vec{x} \in M. i < x_0 \\
&\Leftrightarrow i < \min(\mathsf{pr}_0\, M).
\end{aligned}
$$

Ad 2.,3.: Analog.
Ad 4.:

$$
\begin{aligned}
\{i\} \times \{j\} \times \mathbb{N} \subseteq M^* &\Leftrightarrow \forall k \in \mathbb{N}. \langle i,j,k\rangle \in M^* \\
&\Leftrightarrow \forall k \in \mathbb{N}. \forall \vec{x} \in M. \vec{x} \not\leq \langle i,j,k\rangle \\
&\Leftrightarrow \forall k \in \mathbb{N}. \forall \vec{x} \in M. i < x_0 \vee j < x_1 \vee k < x_2 \\
&\Leftrightarrow \forall \vec{x} \in M. i < x_0 \vee j < x_1
\end{aligned}
$$

Ad 5.,6.: Analog.

⊣

COROLLARY 0.9. *The membership relation for the sets of Definition 0.4 is decidable.*

Next we show that we only need to test a finite set of tuples to count the elements of the "non bar" sets in Definition 0.4.

PROPOSITION 0.10. *Let* $M \in \mathsf{Incomp}_3$ *non empty.*

1. $M_2^0 = \min(\mathsf{pr}_0\, M), M_2^1 = \min(\mathsf{pr}_1\, M), M_2^2 = \min(\mathsf{pr}_2\, M)$
2. $M_1^{0,1} = M_1^{0,1} \cap (\max(\mathsf{pr}_0\, M) \times \max(\mathsf{pr}_1\, M))$
   $M_1^{0,2} = M_1^{0,2} \cap (\max(\mathsf{pr}_0\, M) \times \max(\mathsf{pr}_2\, M))$
   $M_1^{1,2} = M_1^{1,2} \cap (\max(\mathsf{pr}_1\, M) \times \max(\mathsf{pr}_2\, M))$
3. $M_0 = M_0 \cap (\max(\mathsf{pr}_0\, M) \times \max(\mathsf{pr}_1\, M) \times \max(\mathsf{pr}_2\, M))$.

PROOF. Ad 1.: This follows immediatly from the previous Proposition.
Ad 2.: Let $\langle i, j\rangle \in M_1^{0,1}$. We must prove $i < \max(\mathsf{pr}_0\, M)$ and $j < \max(\mathsf{pr}_1\, M)$. We have $i < x_0$ or $j < x_1$ for $\vec{x} \in M$ by the previous Proposition. Since $i \notin \bar{M}_2^0$ exists a $\vec{y} \in M$ with $y_0 \leq i$. It follows

$$j < y_1 \leq \max(\mathsf{pr}_1\, M).$$

Analog follows $i < \max(\mathsf{pr}_0\, M)$ and the claims for the other sets.
Ad 3.: We simply repeat the argument from ad 2. Let $\langle i, j, k\rangle \in M_1^{0,1}$. We must prove $i < \max(\mathsf{pr}_0\, M)$, $j < \max(\mathsf{pr}_1\, M)$ and $k < \max(\mathsf{pr}_2\, M)$. We have $i < x_0$, $j < x_1$ or $k < x_2$ for $\vec{x} \in M$ by the previous Proposition. Since $\langle i, j\rangle \notin \bar{M}_1^{0,1}$ exists a $\vec{y} \in M$ with $y_0 \leq i$ and $y_1 \leq j$. It follows

$$k < y_2 \leq \max(\mathsf{pr}_2\, M).$$

⊣

COROLLARY 0.11. *The functions giving the number of elements of the sets of Definition 0.4 are computable.*

DEFINITION 0.5. *Let* $M \in \mathsf{Incomp}_3$ *non empty.*

$$\mathsf{ord}\, M := \omega^2 \cdot (|M_2^0| + |M_2^1| + |M_2^2|) + \omega \cdot (|M_1^{0,1} + |M_1^{0,2}| + |M_1^{1,2}|) + |M_0|$$

where $|\cdot|$ *denotes the cardinality function.*

We are going to prove that

$$\mathsf{ord}\, M = \sup\{\mathsf{ord}\, N + 1 | N \in \mathsf{Incomp}_3 \wedge N \sqsubset M\}.$$

We will frequently use the following fact:

PROPOSITION 0.12. *Let* $M \in \mathsf{Incomp}_3$ *non empty. The set* M^* *is the union of the following sets*

$$\{\langle i, j, k\rangle \in \mathbb{N}^3 \mid i \in M_2^0\},$$
$$\{\langle i, j, k\rangle \in \mathbb{N}^3 \mid j \in M_2^1\},$$
$$\{\langle i, j, k\rangle \in \mathbb{N}^3 \mid k \in M_2^2\},$$
$$\{\langle i, j, k\rangle \in \mathbb{N}^3 \mid \langle i, j\rangle \in M_1^{0,1}\},$$
$$\{\langle i, j, k\rangle \in \mathbb{N}^3 \mid \langle i, k\rangle \in M_1^{0,2}\},$$
$$\{\langle i, j, k\rangle \in \mathbb{N}^3 \mid \langle j, k\rangle \in M_1^{1,2}\},$$

and M_0.

Further we have

$$\bar{M}_1^{0,1} = M_1^{0,1} \cup \{\langle i,j\rangle \in \mathbb{N}^2 | i \in M_2^0\} \cup \{\langle i,j\rangle \in \mathbb{N}^2 | j \in M_2^1\}$$

etc.

PROOF. Simple verification. ⊣

To shorten our statements we introduce some new notations.

DEFINITION 0.6. *Let $M, N \in \mathsf{Incomp}_3$ non empty.*

$$\begin{aligned}
&N_2 \trianglelefteq M_2 :\Leftrightarrow N_2^0 \subseteq M_2^0 \wedge N_2^1 \subseteq M_2^0 \wedge N_2^2 \subseteq M_2^2\\
&N_1 \trianglelefteq M_1 :\Leftrightarrow N_1^{0,1} \subseteq M_1^{0,1} \wedge N_1^{0,2} \subseteq M_1^{0,2} \wedge N_1^{1,2} \subseteq M_2^{1,2}\\
&N_0 \trianglelefteq M_0 :\Leftrightarrow N_0 \subseteq M_0\\
&N_2 = M_2 :\Leftrightarrow N_2^0 = M_2^0 \wedge N_2^1 = M_2^0 \wedge N_2^2 = M_2^2\\
&N_1 = M_1 :\Leftrightarrow N_1^{0,1} = M_1^{0,1} \wedge N_1^{0,2} = M_1^{0,2} \wedge N_1^{1,2} = M_2^{1,2}\\
&N_i \triangleleft M_i :\Leftrightarrow N_i \trianglelefteq M_i \wedge N_i \neq M_i \text{ for } i = 0,1,2.
\end{aligned}$$

PROPOSITION 0.13. *Let $M, N \in \mathsf{Incomp}_3$ non empty.*

$$N \sqsubseteq M \Rightarrow N_2 \trianglelefteq M_2 \wedge (N_2 = M_2 \Rightarrow N_1 \trianglelefteq M_1) \wedge (N_2 = M_2 \wedge N_1 = M_1 \Rightarrow N_0 \trianglelefteq M_0)$$

PROOF. Let $N \sqsubseteq M$. We have $\forall \vec{x} \in M \exists \vec{y} \in N. \vec{y} \leq \vec{x}$ by Proposition 0.3. Further we have

$$\min(\mathsf{pr}_i\, N) \leq \min(\mathsf{pr}_i\, M) \tag{1}$$

for $i = 0,1,2$: Let $\vec{x} \in M$ with $x_i = \min(\mathsf{pr}_i\, M)$. There is a $\vec{y} \in N$ with $\vec{y} \leq \vec{x}$. It follows

$$\min(\mathsf{pr}_i\, N) \leq y_i \leq x_i = \min(\mathsf{pr}_i\, M).$$

This proves already $N_2 \trianglelefteq M_2$. Further we have

$$\{i\} \times \{j\} \times \mathbb{N} \subseteq N^* \Rightarrow \{i\} \times \{j\} \times \mathbb{N} \subseteq M^* \tag{2}$$

for $i, j \in \mathbb{N}$: We have $\{i\} \times \{j\} \times \mathbb{N} \subseteq N^*$ if and only if $\forall \vec{y} \in N. i < y_0 \vee j < y_1$. Let $\vec{x} \in M$. There is a $\vec{y} \in N$ with $\vec{y} \leq \vec{x}$. It follows

$$i < y_0 \leq x_0 \vee j < y_1 \leq x_1.$$

Therefore $\{i\} \times \{j\} \times \mathbb{N} \subseteq M^*$. This means

$$\bar{N}_1^{0,1} \subseteq \bar{M}_1^{0,1}$$

by Definitions. Analog follows $\bar{N}_1^{0,2} \subseteq \bar{M}_1^{0,2}$ and $\bar{N}_1^{1,2} \subseteq \bar{M}_1^{1,2}$.
Let $N_2 = M_2$ and $\langle i,j\rangle \in N_1^{0,1}$. We have $\langle i,j\rangle \in \bar{N}_1^{0,1} \subseteq \bar{M}_1^{0,1}$, $i \notin \bar{N}_2^0 = \bar{M}_2^0$ and $j \notin \bar{N}_2^1 = \bar{M}_2^1$. Ergo $\langle i,j\rangle \in M_1^{0,1}$ which proves

$$N_1^{0,1} \subseteq M_1^{0,1}.$$

Analog follows $N_1^{0,2} \subseteq M_1^{0,2}$ and $N_1^{1,2} \subseteq M_1^{1,2}$ and therefore

$$N_2 = M_2 \Rightarrow N_1 \trianglelefteq M_1.$$

Finally let $N_2 = M_2$, $N_1 = M_1$. We have

$$\bar{N}_1^{0,1} = \bar{M}_1^{0,1}$$

since

$$\begin{aligned}\bar{N}_1^{0,1} &= N_1^{0,1} \cup \{\langle i,j\rangle \in \mathbb{N}^2 | i \in N_2^0\} \cup \{\langle i,j\rangle \in \mathbb{N}^2 | j \in N_2^1\}\\ &= M_1^{0,1} \cup \{\langle i,j\rangle \in \mathbb{N}^2 | i \in M_2^0\} \cup \{\langle i,j\rangle \in \mathbb{N}^2 | j \in M_2^1\}\\ &= \bar{M}_1^{0,1}\end{aligned}$$

Analog $\bar{N}_1^{0,2} = \bar{M}_1^{0,2}$ and $\bar{N}_1^{1,2} = \bar{M}_1^{1,2}$. Therefore

$$\begin{aligned}N_0 &= N^* \setminus \{\langle i,j,k\rangle \in \mathbb{N}^3 | \langle i,j\rangle \in \bar{N}_1^{0,1} \vee \langle i,k\rangle \in \bar{N}_1^{0,2} \vee \langle j,k\rangle \in \bar{N}_1^{1,2}\}\\ &\subseteq M^* \setminus \{\langle i,j,k\rangle \in \mathbb{N}^3 | \langle i,j\rangle \in \bar{M}_1^{0,1} \vee \langle i,k\rangle \in \bar{M}_1^{0,2} \vee \langle j,k\rangle \in \bar{M}_1^{1,2}\}\\ &= M_0.\end{aligned}$$

⊣

Corollary 0.14. *Let* $M, N \in \mathsf{Incomp}_3$ *non empty.*

$$N \sqsubseteq M \Rightarrow \mathsf{ord}\, N \leq \mathsf{ord}\, M.$$

Proposition 0.15. *Let* $M, N \in \mathsf{Incomp}_3$ *non empty.*

$$N \sqsubset M \Rightarrow N_2 \lhd M_2 \vee N_1 \lhd M_1 \vee N_0 \lhd M_0.$$

Proof. Let $N_2 \not\lhd M_2$ and $N_1 \not\lhd M_1$. By the previous Proposition this means $N_2 = M_2$, $N_1 = M_1$ and $N_0 \unlhd M_0$. Since $N \sqsubset M$ there is an $\langle i,j,k\rangle \in M^* \setminus N^*$. It follows $\langle i,j,k\rangle \in M_0 \setminus N_0$. ⊣

Corollary 0.16. *Let* $M, N \in \mathsf{Incomp}_3$ *non empty.*

$$N \sqsubset M \Rightarrow \mathsf{ord}\, N < \mathsf{ord}\, M.$$

Proposition 0.17. *Let* $M \in \mathsf{Incomp}_3$ *non empty. Then*

$$\mathsf{ord}\, M = \sup\{\mathsf{ord}\, N + 1 | N \in \mathsf{Incomp}_3 \wedge N \sqsubset M\}.$$

Proof. With respect to the previous Corollary it remains to find sets $M(m) \sqsubset M$ in Incomp_3 with

$$\mathsf{ord}\, M = \sup\{\mathsf{ord}\, M(m) + 1 | m \in \mathbb{N}\}.$$

We go through the different cases. Let $\mathsf{ord}\, M = \omega^2 \cdot \alpha_2 + \omega \cdot \alpha_1 + \alpha_0$ and $\mathsf{ord}\, M(m) = \omega^2 \cdot \alpha(m)_2 + \omega \cdot \alpha(m)_1 + \alpha(m)_0$.
First Case: $M_0 \neq \emptyset$ i.e., $\alpha_0 = \alpha_0' + 1$ for some α_0'.
Let $i_0 := \max(\mathsf{pr}_0\, M_0), j_0 := \max(\mathsf{pr}_1\, \{\langle i,j,k\rangle \in M_0 | i_0 = i\}), k_0 := \max(\mathsf{pr}_2\, \{\langle i,j,k\rangle \in M_0 | i = i_0 \wedge j = j_0\})$. Let $M(m) = \mathcal{I}(M \cup \{\langle i_0, j_0, k_0\rangle\})$ for $m \in \mathbb{N}$. We show

$$M^*(m) = M^* \setminus \{\langle i_0, j_0, k_0\rangle\} \tag{3}$$

'$\subseteq$': Let $\vec{x} \in M(m)^*$. By Proposition 0.6 we have $\vec{x} \in (M \cup \{\langle i_0, j_0, k_0\rangle\})^*$. This proves $\forall \vec{y} \in M. \vec{y} \not\leq \vec{x}$ and $\vec{x} \neq \langle i_0, j_0, k_0\rangle$ i.e., $\vec{x} \in M \setminus \{\langle i_0, j_0, k_0\rangle\}$.
'$\supseteq$': Let $\vec{x} \in M^*$ and $\vec{x} \neq \langle i_0, j_0, k_0\rangle$. Let $\vec{y} \in M$. Since $\vec{x} \in M^*$ we have $\vec{y} \not\leq \vec{x}$. Assume $\langle i_0, j_0, k_0\rangle \leq \vec{x}$. Then we have $\langle i_0, j_0, k_0\rangle < \vec{x}$. Further we have $\vec{x} \in M_0$ since from $\langle x_0, x_1\rangle \in \bar{M}_1^{0,1}$ follows $\langle i_0, j_0\rangle \in \bar{M}_1^{0,1}$ etc. But now we got a contradiction since i_0, j_0 and k_0 were choosen maximal with this property.
This proves equation 3. Since the sets giving the coefficients $\alpha(m)_2$ and $\alpha(m)_1$ remain unchanged and we just took one element out of M_0 we have

$$\mathsf{ord}\, M = \mathsf{ord}\, M(m) + 1$$

for $m \in \mathbb{N}$.
Second Case: $M_0 = \emptyset$, $M_1^{0,1} \neq \emptyset$ i.e., $\alpha_0 = 0$ and $\alpha_1 = \alpha_1' + 1$ for some α_1'.
Let $i_0 := \max(\mathsf{pr}_0\, M_1^{0,1})$, $j_0 := \max(\mathsf{pr}_1\, \{\langle i,j\rangle \in M_1^{0,1} | i_0 = i\})$, $k_0 = \max(\mathsf{pr}_2\, M)$.
Let $M(m) = \mathcal{I}(M \cup \{\langle i_0, j_0, k_0 + m\rangle\})$ for $m \in \mathbb{N}$. We show

$$M(m)_2^0 = M_2^0. \tag{4}$$

Since $\langle i_0, j_0\rangle \in M_1^{0,1}$ we have $i_0 \notin M_2^0$ and therefore $i_0 \not< \min(\mathsf{pr}_0\, M)$. Ergo

$$i \in M(m)_2^0 \Leftrightarrow i < \min(\mathsf{pr}_0\, M(m)) = \min(\mathsf{pr}_0\, M) \Leftrightarrow i \in M_2^0.$$

This proves the equation. Analog follows

$$M(m)_2^1 = M_2^1 \tag{5}$$

and by using $\min(\mathsf{pr}_2\, M) \leq k_0$ we can prove in the same way

$$M(m)_2^2 = M_2^2. \tag{6}$$

Next we show

$$\bar{M}(m)_1^{0,1} = \bar{M}_1^{0,1} \setminus \{\langle i_0, j_0\rangle\}.$$

We calculate

$$\begin{aligned} \langle i,j\rangle \in \bar{M}(m)_1^{0,1} &\Leftrightarrow \forall \vec{x} \in M(m). i < x_0 \vee j < x_1 \\ &\Leftrightarrow (\forall \vec{x} \in M. i < x_0 \vee j < x_1) \wedge (i < i_0 \vee j < j_0) \\ &\Leftrightarrow (\forall \vec{x} \in M. i < x_0 \vee j < x_1) \wedge (i \neq i_0 \vee j \neq j_0) \\ &\Leftrightarrow \langle i,j\rangle \in \bar{M}_1^{0,1} \setminus \{\langle i_0, j_0\rangle\} \end{aligned} \tag{7}$$

where the direction from left to right in 7 follows immediate. For the other direction assume $(\forall \vec{x} \in M. i < x_0 \vee j < x_0) \wedge (i \neq i_0 \vee j \neq j_0)$, $i_0 \leq i$ and $j_0 \leq j$. Since $\forall \vec{x} \in M. i < x_0 \vee j < x_0$ we have $\langle i,j\rangle \in \bar{M}_1^{0,1}$. Since $\min(\mathsf{pr}_0\, M) \leq i_0 \leq i$ and $\min(\mathsf{pr}_1\, M) \leq j_0 \leq j$ we have $i \notin M_2^0$ and $j \notin M_2^1$. Therefore $\langle i,j\rangle \in M_1^{0,1}$. But this contradicts i_0 resp. j_0 maximal with this property. We can conclude

$$M(m)_1^{0,1} = M_1^{0,1} \setminus \{\langle i_0, j_0\rangle\}. \tag{8}$$

An analogous calculation as above gives

$$\bar{M}(m)_1^{0,2} = \bar{M}_1^{0,2}$$

where we use

$$(\forall \vec{x} \in M. i < x_0 \vee k < x_2) \wedge (i < i_0 \vee k < k_0) \Leftrightarrow (\forall \vec{x} \in M. i < x_0 \vee k < x_2)$$

instead of 7. The direction from left to right follows again immediate. Since $i_0 \notin \bar{M}_2^0$ there is an $\vec{x} \in M$ with $\vec{x} \leq i_0$. Since $k_0 = \max(\mathsf{pr}_2\, M)$ we also have $x_2 \leq k_0$. Assume $\langle i_0, k_0\rangle \leq \langle i, k\rangle$. Then we have

$$\langle x_0, x_2\rangle \leq \langle i_0, k_0\rangle \leq \langle i, k\rangle$$

and therefore

$$\exists \vec{x} \in M. x_0 \leq i \wedge x_2 \leq k. \tag{9}$$

With the previous calculations we can conclude

$$M(m)_1^{0,2} = M_1^{0,2} \tag{10}$$

and analogous follows

$$M(m)_1^{1,2} = M_1^{1,2}. \tag{11}$$

Finally we understand

$$|m| \leq |M(m)_0|. \tag{12}$$

Therefore we show $\langle i_0, j_0, l\rangle \in M_0(m)$ for $k_0 \leq l < k_0 + m$. First let us see $\langle i_0, j_0, l\rangle \in \bar{M}(m)_0$: Since $\langle i_0, j_0\rangle \in M_1^{0,1}$ we have $\vec{x} \not\leq \langle i_0, j_0, l\rangle$ for all $\vec{x} \in M$ and for $\vec{x} = \langle i_0, j_0, k_0 + m\rangle$ we have $\langle i_0, j_0, l\rangle < \langle i_0, j_0, k_0 + m\rangle$. Therefore $\langle i_0, j_0, l\rangle \in \bar{M}(m)_0$.
On the other side we have $\langle i_0, j_0\rangle \notin \bar{M}(m)_1^{0,1}$ since $\langle i_0, j_0, k_0 + m\rangle \notin M(m)^*$. The existential statement 9 implies $\langle i_0, k_0\rangle \notin \bar{M}(m)_1^{0,2}$ and $\langle j_0, k_0\rangle \notin \bar{M}(m)_1^{1,2}$ can be shown analog. Altogether this gives $\langle i_0, j_0, l\rangle \in M(m)_0$. Summarising the equations 4,5,6,8,10,11 and 12 we receive

$$\mathsf{ord}\, M = \sup\{\mathsf{ord}\, M(m) + 1 | m \in \mathbb{N}\}.$$

The cases $M_0 = M_1^{0,1} = \emptyset$, $M_1^{0,2} \neq \emptyset$ and $M_0 = M_1^{0,1} = M_1^{0,2} = \emptyset$, $M_1^{1,2} \neq \emptyset$ follow analog.
Third case: $M_0 = M_1^{0,1} = M_1^{0,2} = M_1^{1,2} = \emptyset$ and $M_2^0 \neq \emptyset$.
Actually in this case the set M is a singleton i.e., it contains only one element. However we do not use this information and proceed in a way which also applies to higher dimensions.
Let $i_0 := \max(\mathsf{pr}_0\, M_2^0)$, $j_0 := \max(\mathsf{pr}_1\, M)$, $k_0 := \max(\mathsf{pr}_2\, M)$. Let $M(m) = \mathcal{I}(M \cup \{\langle i_0, j_0, k_0 + m\rangle\})$. We have

$$M(m)_2^0 = M_2^0 \setminus \{i_0\} \tag{13}$$

as well as

$$M(m)_2^1 = M_2^1 \quad \text{and} \quad M(m)_2^2 = M_2^2 \tag{14}$$

and

$$|m| \leq |M_1^{0,1}(m)|. \tag{15}$$

The equations 13,14 and 15 prove

$$\mathsf{ord}\, M = \sup\{\mathsf{ord}\, M(m) + 1 | m \in \mathbb{N}\}.$$

The remaining cases follow analog. ⊣

Remark. *For dimensions above* 3 *we lose some of our geometrical intuition. Already the case* $n = 3$ *is hard to visualise. However the geometrical intuition which guided our proof applies as well to higher dimensions. Note how the different cases are treated in a completely similiar way. We can proceed in the same way for* $n > 3$ *.*