Topoi: Theory and Applications

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Meaning

I treat “topos theory” as a theory, whose place is similar to, say, group-theory in relation to semigroup-theory:

1. A “topos” is a special category.
2. Namely a category with “good” “algebraic” structures.
3. Similar to the operations that can be done with finite sets (or arbitrary sets).
4. A “Grothendieck topos” is a special topos, having more “infinitary structure”.
5. It is closer to topology.

Terminology I

- “Topos” is Greek, and means “place”.
- Its use in mathematics likely is close to “space”, either “topological space” or “set-theoretical space”.
- Singular “topos”, plural “topoi” (“toposes” seems to be motivated by a dislike for Greek words).
- Concept invented by Alexander Grothendieck.
- What was original a “topos”, became later a “Grothendieck topos”.
- Grothendieck topoi came from algebraic geometry: “a “topos” as a “topological structure”.
- Giraud (student of Grothendieck) characterised categories equivalent to Grothendieck topoi.

Terminology II

- Then came "elementary topoi", perhaps more motivated from set theory.
- William Lawvere (later together with Myles Tierney) developed "elementary" (first-order) axioms for Grothendieck topoi.
- The "subobject classifier" plays a crucial role here.
- Elementary topoi generalise Grothendieck topoi.
- Nowadays it seems "topos" replaces "elementary topos".

Outline

Generalised topology

From [Borceux, 1994]:

- A **Grothendieck topos** is a category equivalent to a category of sheaves on a site.
- A Grothendieck topos is complete and cocomplete.
- Every Grothendieck topos is a topos.
- A topos in general is only finitely complete and finitely cocomplete.

Three operations with sets

Three related properties of the category $\mathfrak{SET}$:

Function sets For sets A, B we have the set B^A of all maps $f : A \to B$.

Characteristic maps For set A the subsets are in 1-1 correspondence to maps from A to $\{0, 1\}$.

Powersets For a set A we have the set $\mathbb{P}(A)$ of all subsets of A.

Via characteristic maps we get powersets from function spaces:

$$\mathbb{P}(A) \cong \{0, 1\}^A.$$

- Perhaps in categories which have “map objects” and “characteristic maps”, we have also “power objects”?
- Conversely, perhaps from power objects we get, as in set theory, map objects and characteristic maps?

Exponentiation: The idea

Fix a category $\mathfrak{C}$. We consider "exponentiation" with an object E — easiest to fix E:

$$\text{pow}_E : B \in \text{Obj}(\mathfrak{C}) \mapsto B^E \in \text{Obj}(\mathfrak{C}).$$

What could be the universal property? "Currying"?!:

$$\text{Mor}(A \times E, B) \cong \text{Mor}(A, B^E).$$

So we need products in $\mathfrak{C}$. Looks like adjoints?! Recall $F : \mathfrak{C} \to \mathfrak{D}$, $G : \mathfrak{D} \to \mathfrak{C}$ yields an adjoint (F, G) iff there is a natural isomorphism

$$\text{Mor}(F(A), B) \cong \text{Mor}(A, G(B)).$$

So $F(A) := A \times E$ and $G(B) := \text{pow}_E(B)$.

Exponentiation: The formulation I

Definition

The category $\mathfrak{C}$ has exponentiation with power $E \in \mathrm{Obj}(\mathfrak{C})$ if the functor $A \in \mathrm{Obj}(\mathfrak{C}) \mapsto A \times E \in \mathrm{Obj}(\mathfrak{C})$ has a right adjoint $B \in \mathrm{Obj}(\mathfrak{C}) \mapsto B^E \in \mathrm{Obj}(\mathfrak{C})$.

According to the general theory of adjoints, this is equivalent to the property that for all $B \in \mathrm{Obj}(\mathfrak{C})$ the functor $P := (A \times E)_{A \in \mathrm{Obj}(\mathfrak{C})}$ has a universal arrow ("cofree object", "coreflection") from P to B, that is, a morphism

$$e : B^E \times E \to B$$

such that for all $e' : A \times E \to B$ there exists a unique $f : A \to B^E$ with $e' = e \circ (f \times \mathrm{id}_E)$.

Exponentiation: The formulation II

Definition

A category $\mathfrak{C}$ with binary products has exponentiation, if all powers admit exponentiation.

- If $\mathfrak{C}$ has exponentiation, then so does its skeleton, and thus having exponentiation is an invariant under equivalence of categories.
- For exponentiation we need the existence of binary products, however, as usual, a different choice of binary products leads to (correspondingly) isomorphic exponentiations.
- So the above “with” can be interpreted in the weak sense, just sheer existence is enough (no specific product needs to be provided).

Cartesian-closed categories

Definition

A category is cartesian-closed, if it has finite limits and exponentiation.

- Being cartesian-closed is just a property of categories, no additional structure is required ("has" means "exists").
- If a category is cartesian-closed, so is its skeleton, and thus being cartesian-closed is an invariant under equivalence of categories.

The category $\mathfrak{CAT}$ of all (small) categories is cartesian-closed, with $\mathfrak{FUN}(\mathfrak{C}, \mathfrak{D})$ as the exponential object of $\mathfrak{C}, \mathfrak{D}$, and so we can write $\mathfrak{D}^{\mathfrak{C}} := \mathfrak{FUN}(\mathfrak{C}, \mathfrak{D})$.

Subobject classifier: The idea

Now let's turn to "characteristic functions". Consider a category $\mathfrak{C}$ with finite products and an object X.

A subobject A of X shall correspond
to that morphism $\chi_A : X \to \Omega$
for some fixed "subobject classifier" $\Omega \in \mathrm{Obj}(\mathfrak{C})$,
such that the image of A under χ_A
is the same as $\mathrm{t} : 1 \to \Omega$
for some fixed t.

- If such a pair (Ω, t) exists, it is called a "subobject classifier" of $\mathfrak{C}$.
- So consider a subobject(-representation) $i : A \hookrightarrow X$.

Subobject classifier: The conditions

We get the diagram

$$\begin{array}{ccc} A & \xrightarrow{1_A} & 1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[\chi_A]{} & \Omega \end{array}$$

What are the conditions?

1. For every mono $i : A \hookrightarrow X$ there shall be exactly one $\chi_A : X \to \Omega$ making the diagram commute, and fulfilling the further conditions.
2. A pushout?
3. No, a pullback!

Definition

Definition

For a category $\mathfrak{C}$ with a terminal object $1_{\mathfrak{C}}$, a subobject classifier is a pair (Ω, t) with $\Omega \in \mathrm{Obj}(\mathfrak{C})$ and $\mathrm{t} : 1_{\mathfrak{C}} \to \Omega$, such that for all monos $i : A \hookrightarrow X$ there is exactly one $\chi_A : X \to \Omega$ such that $(i, 1_A)$ is a pullback of (t, χ_A).

- All subobject classifiers for $\mathfrak{C}$ are pairwise isomorphic.
- If $\mathfrak{C}$ has a subobject classifier, then so does its skeleton, and thus having a subobject classifier is an invariant under equivalence of categories.

Definition of ("elementary") topoi

Definition

A topos is a cartesian-closed category which has a subobject classifier.

- Being a topos is just a property of categories, no additional structure is required ("has" means "exists").
- If a category is a topos, so is its skeleton, and thus being a topos is an invariant under equivalence of categories.

Equivalent characterisation of topoi

With [Lane and Moerdijk, 1992], Section IV.1:

- The "global" point of view is used, similar to the use of adjoints in characterising exponential objects.
- The "power object" operation $\mathbb{P}$ is assumed.
- This is a map $\mathbb{P} : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Obj}(\mathfrak{C})$ such that for all objects $A, B \in \mathrm{Obj}(\mathfrak{C})$ there are natural isomorphisms

  $$\mathrm{Sub}(A \times B) \cong \mathrm{Mor}(A, \mathbb{P}(B))$$

  (between sets).

Using $\Omega := \mathbb{P}(1)$ we get the subobject classifier.

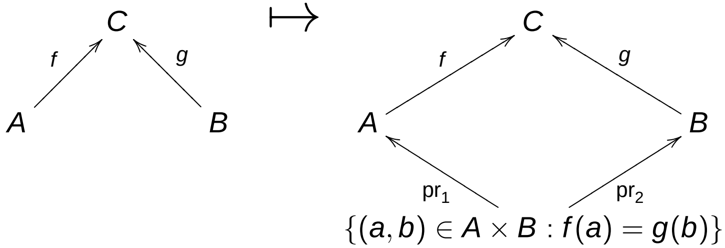
Reminder: limits

- The category of sets is complete, that is, has all (small) limits.
- It is also cocomplete (has all (small) colimits), but we do not need this here (finite cocompleteness follows from being a topos).
- Completeness is equivalent to having all (small) products and all (binary) equalisers.
- The canonical terminal object is the empty product, i.e., $\prod \emptyset = \emptyset^{\emptyset} = \mathbb{P}(\emptyset) = \{\emptyset\} = \{0\} = 1$.

$$(X_i)_{i \in I} \longmapsto \prod_{i \in I} X_i \qquad \ldots \quad X_p \xleftarrow{\mathrm{pr}_p} \prod_{i \in I} X_i \xrightarrow{\mathrm{pr}_q} X_q \quad \ldots$$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \longmapsto \{x \in X : f(x) = g(x)\} \xrightarrow{\mathrm{in}} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Reminder: pullbacks



Exponentiation and subobject classifier

Exponentiation:

$$(X, Y) \longmapsto \boldsymbol{Y^X} := \{f : X \to Y\}$$
$$\boldsymbol{e} : Y^X \times X \to Y, \quad e(f, x) := f(x).$$

Subobject classifier:

$$\boldsymbol{\Omega} := \{0, 1\} = \mathbb{P}(1)$$
$$\boldsymbol{t} : 1 = \{0\} \hookrightarrow \Omega, \quad 0 \mapsto 1.$$

$$\mathbb{P}(X) \hookrightarrow\!\!\!\!\!\rightarrow \Omega^X, \quad A \mapsto \chi_X(A) := A \times \{1\} \cup (X \setminus A) \times \{0\}$$

The category of finite sets

- The full subcategory of $\mathfrak{SET}$ given by all finite sets (in the current universe, of course) is a topos, with the same operations.
- In general, if for a topos $\mathfrak{T}$ and a full subcategory $\mathfrak{C}$
  - $\mathfrak{C}$ is closed under the the topos-operations (finite product, exponentiation, subobject classifier),
  - $\mathfrak{C}$ is closed under subobject-formation,

  then also $\mathfrak{C}$ is a topos.

Monoid operations

Consider a fixed monoid $M = (M, \cdot, 1)$.
An **operation** of M on X is given by a map

$$* : M \times X \to X$$

such that for all $a, b \in M$ and $x \in X$ we have

$$1 * x = x$$
$$a * (b * x) = (a \cdot b) * x.$$

- $(M, *)$ is also called an M-**set.**
- Isomorphically, we have the point of view of a "representation via transformations": a morphism from M into the transformation monoid $\mathfrak{T}(X) = (X^X, \circ, \mathrm{id}_X)$.

See Section 4.6 in [Goldblatt, 2006] for basic information on the topos of M-sets.

The category of M-sets

For M-sets X, Y, a morphism $f : X \to Y$ is a map fulfilling

$$\forall a \in M \, \forall x \in X : f(a * x) = a * f(x).$$

The category of M-sets is denoted by $\boldsymbol{\mathfrak{OPR}_M(\mathfrak{SET})}$.

- I use the terminological distinction between “action” and “operation”, where for the former structure on the object acted upon is involved (e.g., the action of a set on a group via automorphisms), and for the latter structure on the side of acting object (the operation of a group on a set).
- $\mathfrak{OPR}_M(\mathfrak{SET})$ is a concrete category.

$\mathfrak{OPR}_M(\mathfrak{SET})$ is canonically isomorphic to the functor category $\mathfrak{SET}^M$, considering M as a one-object category.

Some remarks

1. The functor $O : \mathfrak{MON} \to \mathfrak{CAT}'$, mapping monoid M to category $\mathfrak{OPR}_M(\mathfrak{SET})$, is a contravariant functor.
2. Here $\mathfrak{CAT}'$ is the category of “large” categories (in the parameter-universe).
3. More generally, the functor $O : \mathfrak{MON} \times \mathfrak{CAT}' \to \mathfrak{CAT}'$, given by $(M, \mathfrak{C}) \mapsto \mathfrak{C}^M$, mapping a monoid M and a category $\mathfrak{C}$ to the category of operations of M on $\mathfrak{C}$, is a bifunctor, contravariant in the first argument.
4. More generally, the mapping $\mathfrak{FUN} : \mathfrak{CAT} \times \mathfrak{CAT}' \to \mathfrak{CAT}'$, $(\mathfrak{C}, \mathfrak{D}) \mapsto \mathfrak{D}^{\mathfrak{C}}$, is a bifunctor, contravariant in the first argument.
5. More generally, for a cartesian-closed category $\mathfrak{C}$, the mapping $\mathfrak{C}^2 \to \mathfrak{C}$, $(X, Y) \mapsto Y^X$, is a bifunctor, contravariant in the first argument.

Limits

The forgetful functor $V : \mathfrak{OPR}_M(\mathfrak{SET})$ has a left-adjoint, the formation of free operations. So V preserves limits.

- I.e., if limits exist, they must have the underlying sets as given by the limits in $\mathfrak{SET}$.
- It is easy to see, as in all algebraic categories, that the operations of M defined in the obvious ways for the $\mathfrak{SET}$-limits, yield limits in $\mathfrak{OPR}_M(\mathfrak{SET})$.

It also follows that the monomorphisms of $\mathfrak{OPR}_M(\mathfrak{SET})$ are precisely the injective morphisms.

Representable functors

The representable functors of a category $\mathfrak{C}$ are those functors $F : \mathfrak{C} \to \mathfrak{SET}$ which are isomorphic to a Hom-functor $X \in \operatorname{Obj} \mathfrak{C} \mapsto \operatorname{Mor}(A, X) \in \operatorname{Obj}(\mathfrak{SET})$ for some $A \in \operatorname{Obj}(\mathfrak{C})$ (the representing object).

- We consider the objects of $\mathfrak{OPR}_M(\mathfrak{SET})$ as functors ("covariant presheafs").
- There is then only one object, thus only one Hom-functor.
- This is the canonical operation of M on itself, via multiplication.

Exponentiation

For M-sets B, E the exponential B^E is defined as having

- base set $\mathrm{Mor}(M \times E, B)$
- operation (for $m \in M$ and a morphism $f : M \times E \to B$)

$$(m * f)(a, e) := f(m \cdot a, e)$$

- evaluation $e : B^E \times E \to B$ given by

$$e(f, e) := f(1, e).$$

Alternative for groups

If M is a group, then we have a simple (of course, isomorphic) possibility to define the exponential B^E:

- base set $\mathrm{Mor}_{\mathfrak{SET}}(E, B)$
- operation (for $g \in M$ and a map $f : E \to B$)

$$(g * f)(e) := g * f(g^{-1} * e)$$

- evaluation $e : B^E \times E \to B$ given by

$$e(f, e) := f(e).$$

Reminder: Yoneda lemma

Lemma

Consider a category $\mathfrak{C}$, an object $A \in \mathrm{Obj}(\mathfrak{C})$ and a functor $T : \mathfrak{C} \to \mathfrak{SET}$. The Yoneda map

$$Y_{A,T} : \mathrm{NAT}(\mathrm{Mor}_{\mathfrak{C}}(A, -), T) \to T(A)$$

is a bijection.

So, for $\mathfrak{C} = \mathfrak{OPR}_M(\mathfrak{SET})$, for every M-set X we have a natural bijection

$$\mathrm{Mor}(M, X) \cong X.$$

This is also easy to see directly, since a morphism from M to X is uniquely determined by the image of 1 (M is the free operation generated by one element).

Subobject classifier

What shall be Ω ?!

Let's consider the M-set M and its subobjects:

1. Subobjects are the left ideals of the semigroup M (subsets stable under left multiplication).
2. As we have seen, $\mathrm{Mor}(M, \Omega) \cong \Omega$ holds.

So we should take as the base set of Ω the set of left ideals of M:

$$\Omega := \{I \subseteq M \mid \forall a \in M \, \forall x \in I : a \cdot x \in I\}.$$

(Thus $|\Omega| \geq 2$.) It is natural to choose $\mathrm{t} := M \in \Omega$.

What is now the operation of M on Ω ?

$$a * \omega := \{b \in M : b \cdot a \in \omega\}$$

for $\omega \in \Omega$ and $a \in M$.

Determining morphisms into Ω

Lemma

For an M-set X and a morphism $f : X \to \Omega$ we have

$$\forall x \in X : f(x) = \{a \in M : f(a * x) = M\}.$$

For a subset $A \subseteq X$ there exists a morphism $f : X \to \Omega$ with $f^{-1}(\{M\}) = A$ iff A is closed (i.e., is a subspace), in which case f is unique, namely $f = \chi_A$ with

$$\chi_A(x) := \{a \in M : a * x \in A\}.$$

Proof: For a morphism $f : X \to \Omega$ we have:

$$f(a * x) = M \Leftrightarrow a * f(x) = M \Leftrightarrow$$
$$\{b \in M : b \cdot a \in f(x)\} = M \Leftrightarrow a \in f(x).$$

M operates trivially on $M \in \Omega$, so $f^{-1}(\{M\})$ is closed. Finally $a * \chi_A(x) = \{b \in M : b \cdot a \in \chi_A(x)\} = \{b \in M : (b \cdot a) * x \in A\} = \{b \in M : b * (a * x) \in A\} = \chi_A(a * x)$. ∎

General directed graphs

Generalisation to functor categories

With Remark B:2.3.19 in [Johnstone, 2002]:

- If $\mathfrak{C}$ is a finite category and $\mathfrak{D}$ is a topos, then $\mathfrak{D}^{\mathfrak{C}}$ is a topos.
- If $\mathfrak{C}$ is a small category and $\mathfrak{D}$ is a cocomplete topos, then $\mathfrak{D}^{\mathfrak{C}}$ is a topos.

For a small category $\mathfrak{C}$
the category $\mathfrak{SET}^{\mathfrak{C}^t}$ of **presheaves**
is a topos.

Reminder: Comma categories

See

`http://en.wikipedia.org/wiki/Comma_category`

for more information.

Consider functors $F : \mathfrak{A} \to \mathfrak{C}$, $G : \mathfrak{B} \to \mathfrak{C}$. The **comma category $(F \downarrow G)$** is defined as follows:

1. objects are triples (a, b, φ), where $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, and $\varphi : F(a) \to G(b)$
2. morphisms $f : (a, b, \varphi) \to (a', b', \varphi')$ are pairs $f = (\alpha, \beta)$, where $\alpha : a \to a'$, $\beta : b \to b'$, and $\varphi' \circ F(\alpha) = G(\beta) \circ \varphi$.

Special cases:

- An object X of a category $\mathfrak{C}$ stands for $1 \mapsto X$.
- A category $\mathfrak{C}$ stands for $\mathrm{id}_{\mathfrak{C}}$.
- $(\mathfrak{C} \downarrow X)$ also written as "$\mathfrak{C}/X$" ("slice category").
- $(\mathfrak{C} \downarrow G)$ also written as "$\mathfrak{C}/G$" ("Artin glueing").

Comma categories yielding topoi

Consider topoi $\mathfrak{B}, \mathfrak{C}$ and a functor $G : \mathfrak{B} \to \mathfrak{C}$.

Theorem [Wraith 1974, [Carboni and Johnstone, 1995]]:

If G preserves pullbacks, then $(\mathfrak{C} \downarrow G)$ *is also a topos.*

Special cases:

1. The product of two topoi is a topos.
2. Slices of a topos are topoi.

The topos of (labelled, generalised) clause-sets

Consider a fixed monoid M.
Consider the (forward) powerset functor

$$\mathbb{P}_{\mathrm{f}} : \mathfrak{OPR}_M(\mathfrak{SET}) \to \mathfrak{SET}$$

which

- maps an M-set X to $\mathbb{P}_{\mathrm{f}}(X)$,
- maps $f : X \to Y$ to $\mathbb{P}_{\mathrm{f}}(f) : \mathbb{P}_{\mathrm{f}}(X) \to \mathbb{P}_{\mathrm{f}}(Y)$, where $\mathbb{P}_{\mathrm{f}}(f)(S) := f(S)$.

Now let

$$\boldsymbol{\mathfrak{LCLS}}_M := (\mathfrak{SET} \downarrow \mathbb{P}_{\mathrm{f}})$$

$\mathbb{P}_{\mathrm{f}}$ does *not* preserve pullbacks, *nevertheless* these categories are topoi.

- $\mathfrak{LCLS}_{\{1\}}$ is the category of labelled hypergraphs
- $\mathfrak{LCLS}_{\mathbb{Z}_2}$ is the category of labelled clause-sets (allowing degenerated and non-polarised literals!).

Without the labelling, we obtain *quasi-topoi*.

A little problem

Actually, [Carboni and Johnstone, 1995] show for topoi $\mathfrak{B}, \mathfrak{C}$ and a functor $G : \mathfrak{B} \to \mathfrak{C}$:

G preserves pullbacks *if and only if* $(\mathfrak{C} \downarrow G)$ is a topos.

Now G clearly does not preserve pullbacks, however we have a topos ...

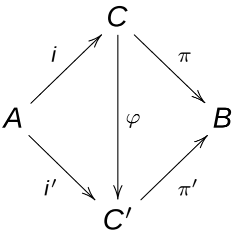
Factorisation

Consider a category $\mathfrak{C}$ and a morphism $f : A \to B$.

- f has an **epi-mono factorisation** if

  $$f = i \circ \pi$$

  for some epimorphism $\pi : A \to C$ and some monomorphism $i : C \to B$.
- Such a factorisation is **unique** if for every other epi-mono factorisation $f = i' \circ \pi'$, $i' : A \to C'$, $\pi' : C' \to B$, there is an isomorphism $\varphi : C \to C'$ with commutative



Topoi have unique factorisations

- A category **has epi-mono factorisation** (also "epi-mono decomposition", or just "factorisation" or "decomposition") if every morphism has an epi-mono factorisation.
- And similarly one says a category **has unique epi-mono factorisation**.

Lemma

A topos has unique epi-mono factorisation.

Topoi are balanced

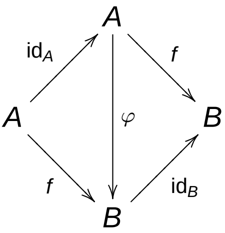
Recall:

- A **bimorphism** is a morphism which is epi and mono.
- A category is **balanced** if every bimorphism is iso.

Lemma

Every category with unique factorisation is balanced.

Proof: Consider a bimorphism $f : A \to B$.



Topoi have power objects

The (global) “power object map” can also be localised.

Topoi are finitely cocomplete

Lemma

Every topos is finitely cocomplete.

Proof: (not completely trivial)

External Heyting algebra

With [Lane and Moerdijk, 1992], Section IV.8:

Lemma

For every object A in a topos, the partial order $\mathrm{Sub}(A)$ *of subobjects is a Heyting lattice.*

Internal Heyting algebras

With [Lane and Moerdijk, 1992], Section IV.8:

Lemma

For every object A in a topos, the power object $\mathbb{P}(A)$ can be given the structure of an Heyting algebra object (an "internal Heyting algebra"). In particular, this applies for the subobject classifier $\Omega = \mathbb{P}(1)$.
For each object X the internal structure of $\mathbb{P}(A)$ makes $\mathrm{Mor}(X, \mathbb{P}(A))$ a Heyting algebra. Now the canonical bijection between $\mathrm{Sub}(X \times A)$ and $\mathrm{Mor}(X, \mathbb{P}(A))$ becomes an isomorphism of Heyting algebras.

Proof: Conjunction $\wedge : \Omega \times \Omega \to \Omega$ is the characteristic morphism of $1 \to \Omega \times \Omega$. ...

Summary

I. The notion of a “topos” has been defined.
II. Examples via categories of presheafs and comma categories have been discussed.
III. Basic elementary properties of topoi have been presented.

Bibliography I

- Borceux, F. (1994).
  Handbook of Categorical Algebra 3: Categories of Sheaves, volume 52 of *Encyclopedia of Mathematics and its Applications*.
  Cambridge University Press.
  ISBN 0 521 44180 3.
- Carboni, A. and Johnstone, P. (1995).
  Connected limits, familial representability and Artin glueing.
  Mathematical Structures in Computer Science, 5(04):441–459.

Bibliography II

Goldblatt, R. (2006).
Topoi: The categorial analysis of logic.
Dover Publications, Inc.
ISBN 0-486-45026-0; slightly corrected unabridged republication of the revised (second) edition originally published in 1984.

Johnstone, P. T. (2002).
Sketches of an Elephant: A Topos Theory Compendium (Volume 1), volume 43 of *Oxford Logic Guides*.
Oxford University Press.
ISBN 0-19-853425-6.

Bibliography III

Lane, S. M. and Moerdijk, I. (1992).
Sheaves in Geometry and Logic: A First Introduction to Topos Theory.
Universitext. Springer.
ISBN 0-387-97710-4; QA169.M335.

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