SAT, Extremal Combinatorics, and elementary Number Theory

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Non-Combinatorial Combinatorics
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Covers of the hypercube (as clause-sets)

The underlying subject of this talk can be seen in the

classification of minimal covers of the hypercube,

in the language of clause-sets (which seems to us more fruitful here).

- For that purpose, we study "four fundamental quantities", which measure the extremal possibilities for parameters related to degrees.
- Upper and lower bounds lead to certain elementary number-theoretic functions.

Since you likely don't know much about that, much of this talk is spent on background.

Is it Non-combinatorial Combinatorics???

When I first registered, I naively thought that outright rejection ("out of scope") yet of any attempt to publish at a combinatorial journal would qualify.

- I realised now that the stakes are higher.
- Combinatorialists think it's not combinatorics since it's SAT.
- Logicians think it's not logic, since I don't use an equivalence relation like logical equivalence, but treat conjunctive normal forms as combinatorial objects.
- What might qualify it for this workshop:
    1. What I have in mind is a "combinatorial SAT" as a "combinatorics of sign patterns" (hypergraphs with a monoid acting on the vertices).
    2. There are relations (mostly hidden in this talk) to the combinatorics of sign patters, and also to algebraic methods.
    3. There are specific methods, which come from the logical background.

Outline

Clause-sets as hypergraphs with complementation

- Let $\mathcal{VA}$ be the set of variables; for concreteness:
$$\mathcal{VA} := \mathbb{N}.$$
- Let $\mathcal{LIT}$ be the set of literals, which are either variables or "complemented" variables. For concreteness:
$$\mathcal{LIT} := \mathbb{Z} \setminus \{0\} = \mathbb{N} \cup -\mathbb{N}.$$
- A clause is a finite and complement-free subset of $\mathcal{LIT}$, the set of all clauses is
$$\mathcal{CL} := \{C \in \mathbb{P}_{\mathrm{f}}(\mathcal{LIT}) : C \cap -C = \emptyset\}.$$
- Let $\mathcal{CLS} := \mathbb{P}_{\mathrm{f}}(\mathcal{CL})$ be the set of clause-sets, finite subsets of $\mathcal{CL}$.

Finally, empty clause and empty clause-set:
$$\begin{aligned} \bot &:= \emptyset \in \mathcal{CL} \\ \top &:= \emptyset \in \mathcal{CLS}. \end{aligned}$$

Minimal unsatisfiability

- A clause-set $F \in \mathcal{CLS}$ is **satisfiable** iff there is a clause $C \in \mathcal{CL}$ which hits every clause of F, the set of all satisfiable clause-sets is

$$\boldsymbol{\mathcal{SAT}} := \{F \in \mathcal{CLS} \mid \exists\, C \in \mathcal{CL}\, \forall\, D \in F : C \cap D \neq \emptyset\}.$$

- $\boldsymbol{\mathcal{USAT}} := \mathcal{CLS} \setminus \mathcal{SAT}$ is the set of **unsatisfiable clause-sets.**
- $F \in \mathcal{USAT}$ is **minimally unsatisfiable** iff for all $C \in F$ holds $F \setminus \{C\} \in \mathcal{SAT}$, the set of all minimally unsatisfiable clause-sets is

$$\boldsymbol{\mathcal{MU}} \subset \mathcal{USAT}.$$

Examples:

1. $\top \in \mathcal{SAT}$
2. $\{\bot\} \in \mathcal{MU}$.
3. $\{\{1\}, \{-1\}\} \in \mathcal{MU}$.

The deficiency

- For $C \in \mathcal{CL}$ let **var(C)** $:= \{|x| : x \in C\} \subset \mathcal{VA}$ be the set of **variables** in C.
- For $F \in \mathcal{CLS}$ let **var(F)** $:= \bigcup_{C \in F} \text{var}(C)$.
- ***n*(*F*)** $:= |\text{var}(F)| \in \mathbb{N}_0$, ***c*(*F*)** $:= |F| \in \mathbb{N}_0$.

For our examples:

1. $n(\top) = c(\top) = 0$.
2. $n(\{\bot\}) = 0$, $c(\{\bot\}) = 1$.
3. $n(\{\{1\}, \{-1\}\}) = 1$, $c(\{\{1\}, \{-1\}\}) = 2$.

A central combinatorial parameter is the ***deficiency*** for $F \in \mathcal{CLS}$:

$$\boldsymbol{\delta(F)} := c(F) - n(F).$$

Theorem (Aharoni and Linial [1])

For $F \in \mathcal{MU}$ holds $\delta(F) \geq 1$.

Hitting clause-sets

A **hitting clause-set** is some $F \in \mathcal{CLS}$ such that each pair of clauses has at least one clash:

$$\begin{aligned} \boldsymbol{\mathcal{HIT}} &:= \{F \in \mathcal{CLS} \mid \forall\, C, D \in F : C \neq D \Rightarrow C \cap -D \neq \emptyset\} \\ \boldsymbol{\mathcal{UHIT}} &:= \mathcal{USAT} \cap \mathcal{HIT}. \end{aligned}$$

In the DNF language, the elements of $\mathcal{HIT}$ are known as “disjoint” or “orthogonal” DNFs. It is easy to see:

$$\mathcal{UHIT} \subset \mathcal{MU}.$$

All our examples have been in $\mathcal{HIT}$ resp. $\mathcal{UHIT}$.
An example of $F \in \mathcal{MU} \setminus \mathcal{UHIT}$ is

$$\big\{ \{1,2\}, \{-1\}, \{-2\} \big\}.$$

Semantics I

"Clause-sets" can be considered as "semantics-free", with two standard interpretations:

CNF the default interpretation, a conjunction of disjunctions: "satisfiable" and "unsatisfiable";

DNF a disjunction of conjunctions: "falsifiable" and "tautology"; "MU" would be "minimal" or "irredundant tautologies".

Semantics II

Closely related to the DNF-interpretation is the interpretation of unsatisfiable clause-sets $F \in \mathcal{USAT}$ as describing coverings of the hypercube $\{0,1\}^n$, where $n = n(F)$:

- The cls-language uses "named variables", while in the hypercube-interpretation "positional variables" are used.
- A clause C describes a sub-cube, via the falsifying/satisfying total assignments in the CNF/DNF-interpretation, with dimension $n - |C|$.
- Note that $n = n(F)$ is the number of actually used dimensions: Our clause-sets do not allow "formal variables" (which do not occur), and the clauses are not degenerated in any sense — different from (general) (hyper)graphs, where you allow vertices not occurring, and where (hyper)edges may have names.
- $\mathcal{MU}$ means minimal/irredundant covering.
- $\mathcal{UHIT}$ means disjoint covering.

The four fundamental parameters I: degrees

For $F \in \mathcal{CLS}$:

- For $v \in \mathcal{VA}$ the **var-degree** is $\mathbf{vd}_F(v) := |\{C \in F : v \in \mathrm{var}(C)\}| \in \mathbb{N}_0$.
- Let $\boldsymbol{\mu}\mathbf{vd}(F) := \min_{v \in \mathcal{VA}} \mathrm{vd}_F(v)$ be the **min-var-degree** of F.

We study for $k \in \mathbb{N}$:

$$\begin{aligned} \mathbf{VDM}(k) &:= \max\{\mu\mathrm{vd}(F) : F \in \mathcal{MU}, \delta(F) = k\} \\ \mathbf{VDH}(k) &:= \max\{\mu\mathrm{vd}(F) : F \in \mathcal{UHIT}, \delta(F) = k\}. \end{aligned}$$

The four fundamental parameters II: full clauses

For $F \in \mathcal{CLS}$:

- For a finite $V \subset \mathcal{VA}$ let $\boldsymbol{A(V)} := \{C \in \mathcal{CL} : \mathrm{var}(C) = V\} \in \mathcal{UHIT}$.
- $\boldsymbol{A(F)} := F \cap A(\mathrm{var}(F))$ is the set of **full clauses** of F.
- $\mathbf{fc}(\boldsymbol{F}) := c(A(F)) \in \mathbb{N}_0$ is the number of full clauses in F.

We study for $k \in \mathbb{N}$:

$$\begin{aligned}\mathbf{FCM}(\boldsymbol{k}) &:= \max\{\mathrm{fc}(F) : F \in \mathcal{MU}, \delta(F) = k\}\\ \mathbf{FCH}(\boldsymbol{k}) &:= \max\{\mathrm{fc}(F) : F \in \mathcal{UHIT}, \delta(F) = k\}.\end{aligned}$$

Hypercube-interpretation I

We are considering minimal resp. disjoint coverings of hypercubes $\{0,1\}^n$ by sub-cubes (i.e., some dimensions fixed to a value); note that n is not fixed here, and does not allow "formal dimensions" (so every minimal covering has at least $n+1$ elements):

1. The deficiency is how more elements are in the cover than n.
2. The degree of a variable is how often the corresponding dimension is fixed.
3. A full clause corresponds to a singleton element in the cover.

Hypercube-interpretation II

So

- VDM(k) resp. VDH(k) is how high you can push the minimum number of "non-$*$s" for the dimensions, in a minimal resp. disjoint covering of deficiency k.
- FCM(k) resp. FCH(k) is the maximal number of singletons in a minimal resp. disjoint covering of deficiency k.

The two number-theoretical functions I

For $n \in \mathbb{N}_0$ let $\boldsymbol{S_2(n)} \in \mathbb{N}_0$ ("Smarandache primitive numbers") be the minimal $k \in \mathbb{N}_0$ with 2^n divides $k!$. Numerical values for $n \geq 0$:

$$0, 2, 4, 4, 6, 8, 8, 8, 10, 12, 12, 14, 16, 16, 16, 16, 18 \ldots$$

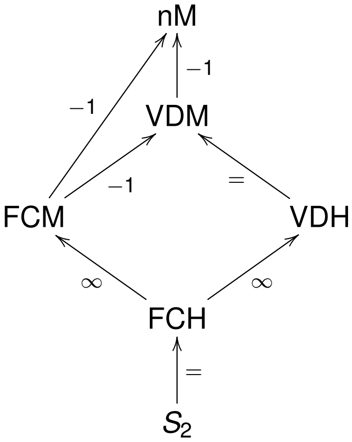
So we enumerate the elements of $\mathbb{N}_0$, taking as many copies of $k \in \mathbb{N}_0$ as prime-factors 2 fit into it.

The two number-theoretical functions II

The map **nM** : $\mathbb{N} \to \mathbb{N}$ ("non-Mersenne numbers") enumerates the element of $\mathbb{N}$, skipping the numbers of the form $2^k - 1$ for $k \in \mathbb{N}$:

$$2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, \ldots$$

Relations between the four quantities



The arrows mean "≤" and are proven (Kullmann and Zhao [8, 9, 10]), the labels are conjectured.

Interlude: Question on efficient algorithms I

An **autarky** (Kleine Büning and Kullmann [7]) for $F \in \mathcal{CLS}$ (here) is a clause C, such that for all $D \in F$ holds $\mathrm{var}(C) \cap \mathrm{var}(D) = \emptyset$ or $C \cap D \neq \emptyset$.

For an autarky for F,
we can remove the clauses satisfied by the autarky from F,
without destroying (possibly) unsatisfiability.

Theorem ([9, Theorem 10.2])

Consider $F \in \mathcal{CLS}$, which after some polytime preprocessing has not already been satisfied (this preprocessing is the algorithmic core of Aharoni and Linial [1]).
Then we can compute in polynomial time some $F' \subseteq F$ with $\mu\mathrm{vd}(F) \leq \mathrm{nM}(\delta(F))$, and where F' is obtained from F by some autarky-reduction.

Interlude: Question on efficient algorithms II

Open Problem

We do not know whether the autarky itself can be computed in polynomial time.

This problem boils down to finding a satisfying assignment for a special class of satisfiable clause-sets.

Initial values

$$\mathcal{SNM} = \{k \in \mathbb{N} : S_2(k) = \mathrm{nM}(k)\}.$$

So for the elements of $\mathcal{SNM}$ we know the four fundamental quantities. The initial elements are:

- $k = 1 \rightsquigarrow 2$:
  1. VDH(1) = 2 was shown in Aharoni and Linial [1].
  2. VDM(1) = 2 was shown in Davydov, Davydova, and Kleine Büning [2], and indeed earlier, in the context of "Qualitative Matrix Analysis", in Klee, Ladner, and Manber [5].
- $k = 2 \rightsquigarrow 4$: VDM(2) = 4 can be easily derived from Kleine Büning [6].

Background: The Classification Conjecture I

A major goal of the research on MU is the project of classifying MU:

Conjecture

For every $k \in \mathbb{N}$ the class $\mathcal{MU}_{\delta=k} = \{F \in \mathcal{MU} : \delta(F) = k\}$ can be characterised by finitely many "patterns" (modulo "basic reductions").

- For $k = 1, 2$ we have good knowledge, but already $k = 3$ is open.
- In Fleischner, Kullmann, and Szeider [3] polytime decision for fixed k was shown.
- As reviewed in [9], the canonical translation of SAT to hypergraph colouring maps $\mathcal{MU}_{\delta=k}$ to critically non-2-colourable hypergraphs with deficiency $k - 1$ (number hyperedges minus number vertices).
- So decision of $\mathcal{MU}_{\delta=1}$ is a special case of the (much more complicated) "even cycle" problem Robertson, Seymour, and Thomas [12], McCuaig [11].

Background: The Classification Conjecture II

An easier open case is the classification of $\mathcal{UHIT}$:

Conjecture

For every deficiency $k \in \mathbb{N}$, after elimination of "singular variables" (appearing in one sign only once) there are only finitely many isomorphism types in $\mathcal{UHIT}_{\delta=k}$.

In other words, for a given deficiency, modulo a simple reduction there are only finitely many types of hypercube partitions.

Recently we have been able to settle the cases $k \leq 3$.

Remarks on methods

The main methods involved in this field are:

- The **resolution operation**, which is a *partial* operation on $\mathcal{CL}$, defined for $C, D \in \mathcal{CL}$ with $|C \cap -D| = \{x\}$ as

$$\boldsymbol{C \diamond D} := (C \setminus \{x\}) \cup (D \setminus \{-x\}).$$

- **Splitting**: For $F \in \mathcal{CLS}$ and $v \in \mathrm{var}(F)$, substitute 0 and 1 into v in F.
- More generally, applying **partial assignments**, which has the structure of a monoid operating in $\mathcal{CLS}$.

Resolution is utilised in many *reductions* (as well as in *extensions*), while splitting is used in *induction*.

A course-of-values recursion for S_2

Theorem ([10])

S_2 fulfils the following recursion schemes (which characterises it uniquely):

$$\begin{aligned} S_2(0) &= 0 \\ S_2(k) &= 2 \cdot (k - \mathrm{i_S}(k) + 1) \end{aligned}$$

for $k \geq 1$, where $\mathrm{i_S}(k)$ is the minimal $i \in \{1, \ldots, k-1\}$ with $k - i + 1 \leq S_2(i)$.

From this we obtain $S_2 \leq \mathrm{FCH}$ (by induction).

A meta-Fibonacci recursion

The key lemma for the proof of the theorem is

Lemma

$\forall\, k \geq 2 : S_2(k) = \sum_{i=1}^{2} S_2(\mathrm{i}_{\mathbb{S}}(k-i))$.

Corollary

$S_2 = 2 \cdot a_2$, where a_2 is from Ruskey and Deugau [13] (derived from Hofstadter [4]), given by

$$a_2(k) = a_2(k - a_2(k-1)) + a_2(k - 1 - a_2(k-2)),$$

while $a_2(k) := k$ for $k \in \{0, 1\}$.

Summary and outlook

I. Generalising from boolean logic to d-valued “logic” for $d \geq 2$ is important for applications, for example for applications to “covering systems”, and we hope to publish this soon.

II. The “four fundamental quantities” seems fascinating objects to us, allowing infinitary levels of improvements of the bounds.

III. The study of them also reveals many useful tools and methods useful for the Classification Conjecture.

End

(references on the remaining slides).

For my papers see
http://cs.swan.ac.uk/~csoliver/papers.html.

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