Reverse Mathematics 2001. edited by Stephen G. Simpson, Lecture Notes in Logic, vol. 21. A K Peters, Ltd., Wellesley, Massachusetts, x + 401 pp.—therein:

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Reverse mathematics is a research program in mathematical logic, pioneered by Harvey Friedman, Stephen Simpson, John Steel, and others in the 1970s. The main question of this program is: *Which set existence axioms are needed to prove the theorems of "ordinary" mathematics?* But reverse mathematics asks for more: one is interested in axioms that are even equivalent to a given theorem, that is, the reverse implication, from the theorem to the axiom, should be provable as well. The first comprehensive account of reverse mathematics was given in Simpson's monograph ***Subsystems of Second Order Arithmetic*** (SOSOA), Perspectives in Mathematical Logic, Springer, 1999. The collected volume under review contains contributions of renowned logicians

and mathematicians to the development of reverse mathematics after SOSOA. Many of the articles present new results within the classical framework of reverse mathematics, but there are also papers that pursue extensions to other areas like higher types or feasible mathematics.

Before describing the contents of the papers in detail, I briefly recall the subsystems of second order arithmetic that are studied most prominently in reverse mathematics. First, there are the systems RCA_0 (Δ_1^0-comprehension plus Σ_1^0-induction), ACA_0 (arithmetic comprehension plus second order induction axiom), and $\Pi_1^1-\mathsf{CA}_0$ (Π_1^1-comprehension plus second order induction axiom). Sandwiched between these systems are WKL_0, which is RCA_0 plus weak König's lemma (every infinite binary tree has an infinite path), and ATR_0, which is ACA_0 plus arithmetical transfinite recursion (a form of transfinite iteration of arithmetical comprehension). We have the proper inclusions $\mathsf{RCA}_0 \subset \mathsf{WKL}_0 \subset \mathsf{ACA}_0 \subset \mathsf{ATR}_0 \subset \Pi_1^1-\mathsf{CA}_0$. It is important to know that the first order part of ACA_0 is PA (Peano arithmetic), while the first order part of WKL_0 is just $\Sigma_1^0-\mathsf{PA}$ (PA with induction restricted to existential formulas). The latter result (due to Friedman) is an important contribution to Hilbert's program as it shows that the system WKL_0, which suffices to develop substantial parts of mathematics, is conservative over a fragment of Peano arithmetic. According to Simpson (SOSOA), the five systems RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi_1^1-\mathsf{CA}_0$ correspond to the foundational programs of constructivism, finitistic reductionism, predicativism, predicative reductionism, and impredicativity, respectively.

In the following, I give an overview of the articles in the volume under review, highlighting some of the results that can be described without going too much into technical details. I arranged the articles into five subject groups: algebra and analysis, infinitary combinatorics, ordinals and comparability, recursion theory and models, feasible mathematics and higher types. Unless stated otherwise, all provability or equivalence results are to be understood relative to the base system RCA_0.

Algebra and analysis. Friedman (10) analyses the following statement about general algebras from the perspective of reverse mathematics: "Every countable algebra with an infinitely generated subalgebra has a maximal infinitely generated subalgebra" ('infinitely generated' is equivalent to 'not finitely generated'). He shows that this statement is equivalent to $\Pi_1^1-\mathsf{CA}_0$ for algebras with one binary function. The same holds for algebras with two unary functions, but for algebras with one unary function the statement is provable in the much weaker theory RCA_0. Downey and Solomon (7) show that Hahn's theorem, a statement about Archimedean subgroups of ordered abelian groups, is equivalent to ACA_0, and hence is not effective. Tanaka and Yamazaki (23) show that a substantial portion of field theory can be done in the weak base theory RCA_0, by proving in RCA_0 the fundamental theorem of algebra as well as quantifier elimination for the theory of real closed fields. Humphrey (15) shows that Cantor's theorem about sets of uniqueness of trigonometric series, which Cantor proved using transfinite ordinals, can in fact be proven in ACA_0, and hence without the use of (advanced) set theory. Brown (3) and Hirst (12) investigate the reverse mathematics of compactness. Brown shows, for example, that the statement "every totally bounded complete separable metric space is Heine-Borel compact" is equivalent to the system WKL_0, while Hirst answers affirmatively Friedman's question whether the equivalence between the Heine-Borel theorem and WKL_0 (proven by Friedman) still holds if the Heine-Borel theorem is restricted to the rationals. The reviewer remarks that the reverse mathematics of compactness is also being studied within the framework of Bishop's constructive mathematics, where the role of weak König's lemma is taken over by its contrapositive, the fan theorem (see, for example, Ishihara, ***Constructive Reverse Mathematics: Compactness Properties***, in: ***From Sets and Types to***

Topology and Analysis, Laura Crosilla and Peter Schuster, editors, Oxford Logic Guides 48, Clarendon Press, Oxford, 2005, pp. 245–267).

Infinitary combinatorics. Cenzer and Remmel (4) establish equivalences of various versions of the stable marriage problem (the finite version is a classic problem in graph theory) to ACA_0, ATR_0, and WKL_0. They also prove the equivalence of WKL_0 to various combinatorial results on posets, for example Dilworth's theorem: any poset of width k can be covered by k chains. The paper of Marcone (19) gives a survey of the reverse mathematics of WQO (well quasi ordering) and BQO (better quasi ordering) theory and contains some new results about the proof theoretic strength of closure properties of well quasi orderings. WQO and BQO theory is of particular interest to reverse mathematics since it is a rich source of natural mathematical theorems that require strong set existence axioms for their proofs (for example, the graph minor theorem is not provable in $\Pi_1^1-\mathsf{CA}_0$, as shown by Friedman, Robertson and Seymour). Schmerl (20) proves theorems about (non) k-colorability of infinite graphs in the theory $\mathsf{RCA}_0+\neg\mathsf{WKL}$ (which has the recursive sets as a model). As a technical tool he uses the interesting game theoretic concept of on-line-coloring. The paper by Cholak, Giusto, Hirst, and Jokusch (5) investigates the reverse mathematics of statements related to Ramsey's theorem, namely the free set theorem and a weakening thereof, the thin set theorem. It is shown that the free set theorem has a similar strength to Ramsey's theorem. Some relationships between these theorems are proven and some questions are formulated as open problems.

Ordinals and comparability. Hirst (13) surveys the reverse mathematics of ordinal arithmetic. For example, ACA_0 is equivalent to arithmetical transfinite induction, and ATR_0 is equivalent to the comparability of well orderings. Friedman (11) strengthens some of these results and proves similar comparability results for countable metric spaces. Hirst (14) also shows that ATR_0 is equivalent to the so-called γ-lemma, stating that $\Sigma_{\alpha<\omega^\gamma} f(\alpha) = \sup\{f(\alpha)\cdot\omega^\gamma \mid \alpha < \omega^\gamma\}$ for every nondecreasing function f from ω^γ to the ordinals. If f is the identity, one obtains Sierpinski's exercise ($\Sigma_{\alpha<\omega^n}\alpha = \omega^{2n-1}$) which is already provable in RCA_0 (here, of course, α, γ range over countable well orderings and the equality sign has to be interpreted appropriately).

Recursion theory and models. Knight (16) and Arana (1) study the Turing degrees of m-diagrams of models of Peano arithmetic. Knight's main result is that to every nonstandard model of PA there exists an isomorphic model whose m-diagrams form a strict chain with respect to Turing reducibility. Arana characterizes the degrees of m-diagrams of models of true arithmetic and other models of PA. Chong, Shore, and Yang (6) translate formulas φ in the language of PA into formulas φ^* in the language of the partial order of recursively enumerable degrees such that in any model of Σ_4-induction φ holds iff φ^* holds in the recursively enumerable degrees of that model. Kossak (18) argues that arithmetic saturation relates to recursive saturation as ACA_0 relates to WKL_0. He proves the following result in reverse model theory: the statement that every consistent theory in a countable language has an arithmetically saturated model is equivalent to ACA_0^+, where ACA_0^+ extends ACA_0 by a scheme of ω-times iterated arithmetical comprehension. Schmerl (21) introduces the notion of a $\Delta_{1,n}$ set, that is, a set of natural numbers whose characteristic function is the limit of an n-trial-and-error-function. He uses $\Delta_{1,n}$ sets to give a new proof of Jokusch's result that PA has no completion in the set of boolean combinations of recursively enumerable sets. Interpreting the notion of a $\Delta_{1,n}$ set in the context of RCA_0, he also extends Simpson's classic result in reverse mathematics about the equivalence of WKL_0 to Lindenbaum's lemma stating that every countable consistent set has a completion. Schmerl shows that the equivalence still holds if Lindenbaum's lemma is formulated for completions in $\Delta_{1,n}$. Simpson's contribution (22) addresses a central foundational problem in analysis,

namely the fact that important theorems of classical analysis fail in computable analysis (where only computable reals are considered). For example, the maximum principle for continuous real functions on the unit interval, which is equivalent to WKL_0, fails in computable analysis. Using the theory of Medvedev degrees and ideas from forcing, Simpson shows that there is a countable ω-model of WKL_0 such that every definable set that exists in the model is recursive. Hence recursive and classical analysis can be reconciled to a certain extent.

Feasible mathematics and higher types. The remaining five papers of this collection represent developments in reverse mathematics beyond the classical framework of subsystems of second order arithmetic between RCA_0 and $\Pi_1^1-\mathsf{CA}_0$. The papers by Fernandes, Ferreira, and Yamazaki are set within Ferreira's theory BTFA of basic feasible analysis whose provable recursive functions are all polynomial time computable. Fernando and Ferreira (9) show that the Heine-Borel theorem is equivalent to $\Pi_1^b-\mathsf{WKL}$, a version of weak König's lemma for trees defined by Π_1^b-formulas (formulas with one binary length bounded universal quantifier). They also show that the statement "every real function on the unit interval is uniformly continuous" lies between WKL and $\Pi_1^b-\mathsf{WKL}$. Fernando (8) uses a forcing technique to show that the Baire category theorem is conservative (over BTFA) for sentences of the form $\forall X\ \exists! Y \varphi(X,Y)$ with arithmetical φ. Yamazaki (24) proves that, for real functions on the unit interval, the intermediate value theorem is provable (in BTFA) and a version of the maximum principle is equivalent to Σ_1^b-comprehension. The articles by Avigad and Kohlenbach have in common that they both extend the universe of discourse from natural numbers and sets of natural numbers to functionals of finite types over the natural numbers (sets of natural numbers are represented by their characteristic functions). Higher type systems are attractive because they are at the same time expressive and proof theoretically weak (often conservative over their first order parts). Avigad (2) uses a forcing technique to obtain nonstandard extensions of weak systems of arithmetic in higher types, including polynomial-time computable arithmetic. He proves a number of conservation results and describes how to formalize weak systems of analysis in such systems. Kohlenbach (17) stresses the need for higher types to express e.g. statements about non-continuous functions. He sketches a development of reverse mathematics in higher types where set existence principles are replaced by existence principles for higher type functionals. One of his results in this direction is, for example, the pairwise equivalence (over a suitable higher type version of RCA_0) of the uniform weak König's lemma, the uniform intermediate value theorem, and the existence of the type 2 functional $\exists^2$ representing quantification over numbers.

I found this book very inspiring and enjoyable. The articles give a broad account of the state of the art in reverse mathematics and highlight interesting new developments in current research on Hilbert's program. The book is therefore essential reading for logicians and philosophers working in the area of the foundations of mathematics. All articles are clearly written and well motivated. Most of them are also fairly self contained and provide short introductions into the formal frameworks they use. Therefore, this book is accessible and of interest not only to specialists, but also to "ordinary" mathematicians with a basic logic background.

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